REFERENCES

[CW] R. R. Coifman and Y. Meyer, *Nouvelles bases othonormées de $L^2(\mathbb{R})$ ayant la structure du système de Walsh*, preprint, Yale University, New Haven (1989).

[D] Ingrid Daubechies, *Orthonormal bases of compactly supported wavelets*, Communications on Pure and Applied Mathematics **XLI** (1988), 909–996.

[Ma] Stephane G. Mallat, *A Theory for Multiresolution Signal Decomposition: The Wavelet Decomposition*, IEEE Transactions on Pattern Analysis and Machine Intelligence **11** (1989), 674–693.

Appendix III

Nonstandard Matrix Multiplication

Wave packets. Define wave packets over l^2 in the usual way. For a pair $P = \{p_i\}, Q = \{q_i\}$ of quadrature mirror filters (QMFs) satisfying the orthogonality and decay conditions stated in [CW], there is a unique solution to the functional equation

$$\phi(t) = \sqrt{2}\sum_{j\in\mathbf{Z}} p_j \phi(2t-j).$$

Put $w = w_{0,0,0} = \phi$, and define recursively

$$w_{2n,0,0}(t) = \sqrt{2}\sum_{j\in\mathbf{Z}} p_j w_{n,0,0}(2t-j),$$
$$w_{2n+1,0,0}(t) = \sqrt{2}\sum_{j\in\mathbf{Z}} q_j w_{n,0,0}(2t-j).$$

Then set $w_{nmk}(t) = 2^{m/2} w_{n00}(2^m t - k)$. Write $\mathcal{W}(\mathbf{R}) = \{w_{nmk} : n, m, k \in \mathbf{Z}\}$ for the collection of functions so defined, which we shall call *wave packets.*

The quadrature mirror filters P, Q may be chosen so that $\mathcal{W}(\mathbf{R})$ is dense in many common function spaces. With the minimal hypotheses of [CW], $\mathcal{W}(\mathbf{R})$ will be dense in $L^2(\mathbf{R})$. Using the Haar filters $P = \{1/\sqrt{2}, 1/\sqrt{2}\}, Q = \{1/\sqrt{2}, -1/\sqrt{2}\}$ produces $\mathcal{W}(\mathbf{R})$ which is dense in $L^p(\mathbf{R})$ for $1 < p < \infty$. Longer filters can generate smoother wave packets, so we can also produce dense subsets of Sobolev spaces, etc.

Basis subsets. Define a *basis subset* σ of the set of indices $\{(n, m, k) \in \mathbf{Z}^3\}$ to be any subcollection with the property that $\{w_{nmk} : (n, m, k) \in \sigma\}$ is a Hilbert basis for $L^2(\mathbf{R})$. We characterize basis subsets in [W1]. Abusing notation, we shall also refer to the collection of wave packets $\{w_{nmk} : (n, m, k) \in \sigma\}$ as a basis subset.

Since $L^2 \cap L^p$ is dense in L^p for $1 \le p < \infty$, with certain QMFs a basis subset will also be a basis for L^p. Likewise, for nice enough QMFs, it will be a Hilbert basis for the various Sobolev spaces.

Since $L^2(\mathbf{R}) \otimes L^2(\mathbf{R})$ is dense in $L^2(\mathbf{R}^2)$, the collection of vectors $\{w_X \otimes w_Y : w_X \in \mathcal{W}(X), w_Y \in \mathcal{W}(Y)\}$ is dense in the space of Hilbert-Schmidt operators. Call $\sigma \subset \mathbf{Z}^6$ a

III-2

basis subset if the collection $\{w_{n_X m_X k_X} \otimes w_{n_Y m_Y k_Y} : (n_X, m_X, k_X, n_Y, m_Y, k_Y) \in \sigma\}$ forms a Hilbert basis. Such two-dimensional basis subsets are characterized in [W2].

Ordering wave packets. Wave packets w_{nmk} can be totally ordered. We say that $w < w'$ if $(m, n, k) < (m', n', k')$. The triplets are compared lexicographically, counting the scale parameter m as most significant.

Tensor products of wave packets inherit this total order. Write $w_X = w_{n_X m_X k_X}$, etc. Then we will say that $w_X \otimes w_Y < w'_X \otimes w'_Y$ if $w_X < w'_X$, or else if $w_X = w'_X$ but $w_Y < w'_Y$. This is equivalent to

$$(m_X, n_X, k_X, m_Y, n_Y, k_Y) < (m'_X, n'_X, k'_X, m'_Y, n'_Y, k'_Y),$$

comparing lexicographically from left to right.

Define the *adjoint order* $<^*$ by exchanging X and Y indices, namely $w_X \otimes w_Y <^* w'_X \otimes w'_Y$ if and only if $w_Y \otimes w_X <^* w'_Y \otimes w'_X$. This is also a total order.

Projections. Let $\mathcal{W}^1$ denote the space of bounded sequences indexed by the three wave packet indices n, m, k. With the ordering above, we obtain a natural isomorphism between l^∞ and $\mathcal{W}^1$. There is also a natural injection $J^1 : L^2(\mathbf{R}) \hookrightarrow \mathcal{W}^1$ given by $c_{nmk} = \langle v, w_{nmk}\rangle$, for $v \in L^2(\mathbf{R})$ and $w_{nmk} \in \mathcal{W}(\mathbf{R})$. If σ is a basis subset, then the composition J^1_σ of J^1 with projection onto the subsequences indexed by σ is also injective. J^1_σ is an isomorphism of $L^2(\mathbf{R})$ onto $l^2(\sigma)$, which is defined to be the square summable sequences of $\mathcal{W}^1$ whose indices belong to σ.

We also have a map $R^1 : \mathcal{W}^1 \to L^2(\mathbf{R})$ defined by

$$R^1 c(t) = \sum_{(n,m,k)\in \mathbf{Z}^3} c_{nmk} w_{nmk}(t).$$

This map is defined and bounded on the closed subspace of $\mathcal{W}^1$ isomorphic to l^2 under the natural isomorphism mentioned above. In particular, R^1 is defined and bounded on the range of J^1_σ for every basis subset σ. The related restriction $R^1_\sigma : \mathcal{W}^1 \to L^2(\mathbf{R})$ defined by $R^1_\sigma c(t) = \sum_{(n,m,k)\in\sigma} c_{nmk} w_{nmk}(t)$ is a left inverse for J^1 and J^1_σ. In addition, $J^1 R^1_\sigma$ is a projection of $\mathcal{W}^1$. Likewise, if $\sum_i \alpha_i = 1$ and $R^1_{\sigma_i}$ is one of the above maps for each i, then $J^1 \sum_i \alpha_i R^1_{\sigma_i}$ is also a projection of $\mathcal{W}^1$. It is an orthogonal projection on any finite subset of $\mathcal{W}^1$.

Similarly, writing $\mathcal{W}^2$ for $\mathcal{W}^1 \times \mathcal{W}^1$, the ordering of tensor products gives a natural isomorphism between l^∞ and $\mathcal{W}^2$. The space $L^2(\mathbf{R}^2)$, i.e., the Hilbert-Schmidt operators, inject into this sequence space $\mathcal{W}^2$ in the obvious way, namely $M \mapsto$

$\langle M, w_{n_X m_X k_X} \otimes w_{n_Y m_Y k_Y} \rangle$. Call this injection J^2. If σ is a basis subset of $\mathcal{W}^2$, then the composition J^2_σ of J^2 with projection onto subsequences indexed by σ is also injective. J^2_σ is an isomorphism of $L^2(\mathbf{R}^2)$ onto $l^2(\sigma)$, the square summable sequences of $\mathcal{W}^2$ whose indices belong to σ.

The map $R^2 : \mathcal{W}^2 \to L^2(\mathbf{R}^2)$ given by $R^2 c(x,y) = \sum c_{XY} w_X(x) w_Y(y)$, is bounded on that subset of $\mathcal{W}^2$ naturally isomorphic to l^2. In particular, it is bounded on the range of J^2_σ for every basis subset σ.

We may also define the restrictions R^2_σ of R^2 to subsequences indexed by σ, defined by $R^2_\sigma c(x,y) = \sum_{(w_X, w_Y)\in\sigma} c_{XY} w_X(x) w_Y(y)$. There is one for each basis subset σ of $\mathcal{W}^2$. Then R^2_σ is a left inverse of J^2 and J^2_σ, and $J^2 R^2_\sigma$ is a projection of $\mathcal{W}^2$. As before, if $\sum_i \alpha_i = 1$ and σ_i is a basis subset for each i, then $J^2 \sum_i R^2_{\sigma_i}$ is also a projection of $\mathcal{W}^2$. It is an orthogonal projection on any finite subset of $\mathcal{W}^2$.

Applying operators to vectors. For definiteness, let X and Y be two named copies of $\mathbf{R}$. Let $v \in L^2(X)$ be a vector, whose coordinates with respect to wave packets form the sequence $J^1 v = \{\langle v, w_X\rangle : w_X \in \mathcal{W}(X)\}$.

Let $M : L^2(X) \to L^2(Y)$ be a Hilbert-Schmidt operator. Its matrix coefficients with respect to the complete set of tensor products of wave packets form the sequence $J^2 M = \{\langle M, w_X \otimes w_Y\rangle : w_X \in \mathcal{W}(X), w_Y \in \mathcal{W}(Y)\}$. We obtain the identity

$$\langle Mv, w_Y\rangle = \sum_{w_X \in \mathcal{W}(X)} \langle M, w_X \otimes w_Y\rangle \langle v, w_X\rangle$$

This identity generalizes to a linear action of $\mathcal{W}^2$ on $\mathcal{W}^1$ defined by

$$c(v)_{nmk} = \sum_{(n'm'k')} c_{nmkn'm'k'} v_{n'm'k'}.$$

Now, images of operators form a proper submanifold of $\mathcal{W}^2$. Likewise, images of vectors form a submanifold $\mathcal{W}^1$. We can lift the action of M on v to these larger spaces via the commutative diagram

$$\begin{array}{ccc} \mathcal{W}^1 & \xrightarrow{J^2_\sigma M} & \mathcal{W}^1 \\ {\scriptstyle J^1}\uparrow & & \downarrow{\scriptstyle R^1} \\ L^2(\mathbf{R}) & \xrightarrow[M]{} & L^2(\mathbf{R}) \end{array}$$

The significance of this lift is that by a suitable choice of σ we can reduce the complexity of the map $J^2_\sigma M$, and therefore the complexity of the operator application.

Composing operators. Let X, Y, Z be three named copies of $\mathbf{R}$. Suppose that $M : L^2(X) \to L^2(Y)$ and $N : L^2(Y) \to L^2(Z)$ are Hilbert-Schmidt operators. We have the identity

$$\langle NM, w_X \otimes w_Z \rangle = \sum_{w_Y \in \mathcal{W}(Y)} \langle N, w_Y \otimes w_Z \rangle \langle M, w_X \otimes w_Y \rangle.$$

This generalizes to an action of $\mathcal{W}^2$ on $\mathcal{W}^2$, which is defined by the formula

$$c(d)_{nmkn'm'k'} = \sum_{n''m''k''} d_{nmkn''m''k''} c_{n''m''k''n'm'k'},$$

where c and d are sequences in $\mathcal{W}^2$. Using J^2, we can lift multiplication by N to an action on these larger spaces via the commutative diagram

$$\begin{tikzcd} \mathcal{W}^2 \arrow[r, "J^2_\sigma N"] & \mathcal{W}^2 \arrow[d, "R^2"] \\ L^2(\mathbf{R}^2) \arrow[u, "J^2"] \arrow[r, "N"'] & L^2(\mathbf{R}^2) \end{tikzcd}$$

Again, by a suitable choice of σ the complexity of the operation may be reduced to below that of ordinary operator composition.

Operation counts: transforming a vector. Suppose that M is a non-sparse operator of rank r. Ordinary multiplication of a vector by M takes at least $O(r^2)$ operations, with the minimum achievable only by representing M as a matrix with respect to the bases of its r-dimensional domain and range.

On the other hand, the injection J^2 will require $O(r^2[\log r]^2)$ operations, and each of J^1 and R^1 require $O(r \log r)$ operations. For a fixed basis subset σ of $\mathcal{W}^2$, the application of $J^2_\sigma M$ to $J^1 v$ requires at most $\#|J^2_\sigma M|$ operations, where $\#|U|$ denotes the number of nonzero coefficients in U. We may choose our wavelet library so that $\#|J^2_\sigma M| = O(r^2)$. Thus the multiplication method described above costs an initial investment of $O(r^2[\log r]^2)$, plus at most an additional $O(r^2)$ per right-hand side. Thus the method has asymptotic complexity $O(r^2)$ per vector in its exact form, as expected for what is essentially multiplication by a conjugated matrix.

We can obtain lower complexity if we take into account the finite accuracy of our calculation. Given a fixed matrix of coefficients C, write C_δ for the same matrix with all coefficients set to 0 whose absolute values are less than δ. By the continuity of the Hilbert-Schmidt norm, for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|C - C_\delta\|_{HS} < \epsilon$. Given

M and ϵ as well as a library of wave packets, we can choose a basis subset $\sigma \subset \mathcal{W}^2$ so as to minimize $\#|(J^2_\sigma M)_\delta|$. The choice algorithm has complexity $O(r^2[\log r]^2)$, as shown in [W2]. For a certain class of operators, there is a library of wave packets such that for every fixed $\delta > 0$ we have

$$\text{(S)} \qquad \#|(J^2_\sigma M)_\delta| = O(r \log r),$$

with the constant depending, of course, on δ. We will characterize this class, give examples of members, and find useful sufficient conditions for membership in it. For the moment, call this class with property S the *sparsifiable* Hilbert-Schmidt operators $\mathcal{S}$. By the estimate above, finite-precision multiplication by sparsifiable rank-r operators has asymptotic complexity $O(r \log r)$.

Operation counts: composing two operators. Suppose that M and N are rank-r operators. Standard multiplication of N and M has complexity $O(r^3)$. The complexity of injecting N and M into $\mathcal{W}^2$ is $O(r^2[\log r]^2)$. The action of $J^2_\sigma N$ on $J^2 M$ has complexity $O(\sum_{nmk} \#|J^2_\sigma N_{YZ} : (n_Y, m_Y, k_Y) = (n, m, k)|\#|J^2 M_{XY} : (n_Y, m_Y, k_Y) = (n, m, k)|)$. The second factor is a constant $r \log r$, while the first when summed over all nmk is exactly $\#|J^2_\sigma N|$. Thus the complexity of the nonstandard multiplication algorithm, including the conjugation into the basis set σ, is $O(\#|J^2_\sigma N| r \log r)$. Since the first factor is r^2 in general, the complexity of the exact algorithm is $O(r^3 \log r)$ for generic matrices, reflecting the extra cost of conjugating into the basis set σ.

For the approximate algorithm, the complexity is $O(\#|(J^2_\sigma N)_\delta| r \log r)$. For the sparsifiable matrices, this can be reduced by a suitable choice of σ to a complexity of $O(r^2[\log r]^2)$ for the complete algorithm. Since choosing σ and evaluating J^2_σ each have this complexity, it is not possible to do any better by this method.

Reference

[CW] –, *Appendix II.*
[W2] –, *Appendix V.*

Appendix IV

Entropy of a Vector Relative to a Decomposition

Let $v \in H$ $\|v\| = 1$ and assume

$$H = \oplus \sum H_i$$

an orthogonal direct sum. We define

$$\varepsilon^2(v, \{H_i\}) = -\sum \|v_i\|^2 \ell u \|v_i\|^2$$

as a measure of distance between v and the orthogonal decomposition.

ε^2 is characterized by the Shannon equation which is a version of Pythagoras' theorem.

Let

$$\begin{aligned} H &= \oplus(\sum H^i) \oplus (\sum H_j) \\ &= H_+ \oplus H_- \end{aligned}$$

H^i and H_j give orthogonal decomposition $H_+ = \sum H^i$ $H_- = \sum H_j$. Then

$$\varepsilon^2(v; \{h^i, h_j\}) = \varepsilon^2(v, \{H_+, H_-\} + \|v_+\|^2 \varepsilon^2 \left(\frac{v_+}{\|v++\|}, \{H^i\}\right) + \|v_-\|^2 \varepsilon^2 \left(\frac{v_-}{\|v\|}, \{H_j\}\right)$$

This is Shannon's equation for entropy (if we interpret as in quantum mechanics $\|P_{H_+} v\|^2$ as the "probability" of v to be in the subspace H_+).

This equation enables us to search for a smallest entropy space decomposition of a given vector. We need the following $H = H_1 \oplus H_2$.

$$H_1 = \oplus \sum H^i = \oplus \sum K^j.$$

H_1 has two decompositions in H^i or K^j.

Lemma 1. *Let $v \in H$ with $\|v\|$ and*

$$v_1 = P_{H_1}v \qquad v_1' = \frac{v_1}{\|v_1\|}$$

Assume also that $\varepsilon^2(v_1', \{H^i\}) < \varepsilon^2(v_1', \{K^i\})$ then, if $H_2 = \oplus \sum L^j$ we have

$$\varepsilon^2(v, \{H^i, L^j\}) < \varepsilon^2(v, \{K^j, L^i\})$$

Proof. By Shannon's equation

$$\begin{aligned}
\varepsilon^2(v, \{H^i, L^j\}) &= \varepsilon^2(v, \{H_1, H_2\}) + \|v_1\|^2\varepsilon^2(v_1', \{H^i\}) \\
&\quad + \|v_2\|^2\varepsilon^2(v_2', \{L^j\}) \\
&< \varepsilon^2(v, \{H_1, H_2\}) + \|v_1\|^2\varepsilon^2(v_1', \{K^j\}) + \|v_2\|^2\varepsilon^2(v_2', \{L^j\}) \\
&= \varepsilon^2(v, \{K^j, L^i\}).
\end{aligned}$$

Corollary. *Assume $\varepsilon^2(v_1', \{H^i\})$ is the smallest entropy obtained for some collection of decompositins of H_1 and similarly, $\varepsilon^2(v_2', \{L^j\})$ is minimal. Then $\varepsilon^2(v; \{H^i, L^j\})$ is minimal for the direct sum of these collections.*

We consider the following generic example on $L^2(\mathbf{R}_+)$.

Let I denote a dyadic interval of the form $(2^j k, 2^j(k+1)]$, $j \geq 0$ $k \geq 0$, and $\{I_\alpha\}$ a disjoint cover over $(0, \infty)$ consisting of dyadic intervals. We let $H_{I_\alpha} = L^2(I_\alpha)$ on which we chose an orthonormal basis $\{e^{I_\alpha}_{\alpha,k}\}$ α fixed (say trig polynomials $\exp(2\pi i \frac{x}{2^j})\chi_{I_\alpha}(x)$.) and consider $\{e^{I_\alpha}_{\alpha,k}\}$ as an orthonormal basis of $L^2(\mathbf{R}^+)$. Thus

$$L^2(\mathbf{R}^+ = \sum H_{I_\alpha} = \sum_\alpha \sum_k \{e_{\alpha,k}\}$$

Given a vector v we wish to find I_α such that $\varepsilon^2(v, \{e_{\alpha,k}\})$ is minimal. In order to find I_α we use a stopping time argument. Starting with intervals of length one $I_\ell = (\ell, \ell+1]$. We pick a dyadic interval of length two which contains halves $I_1 I_2$ of length one, i.e. $J = I_1 \cup I_2$. We compart

$$\varepsilon^2(v\chi_J, \{e^J_k\}) \quad \text{with} \quad \varepsilon^2(v\chi_J, \{e^{I_1}_k\}\{e^{I_2}_k\})$$

and pick the basis given the smallest entropy leading to a cover of $L^2(\mathbf{R})$ by intervals of length one and two. We now consider dyadic intervals K of length 4 and compare

$$\varepsilon^2(v\chi_K\{e^K_j\}) \quad \text{with} \quad \varepsilon^2(v\chi_K; \{e^{I_\alpha}_K\})$$

where I_α form a cover of K by dyadic intervals of length one or two selected previously to minimize ε^2 on each half of K.

(If the vector function v has bounded support we restrict our attention only to dyadic intervals contained in the smallest dyadic interval containing the support of v) and continue this procedure up to this largest scale. We claim that the final partition I_α and corresponding basis provides the minimal entropy decomposition. In fact, this is an immediate consequence of Lemma 1 which shows that given the optimal minimum entropy partition any refinement corresponds to worse entropy.

Appendix V

Higher-Dimensional Best Basis Selection

Introduction. We introduce a method of coding by orthogonal functions which may be used to compress digitized pictures or sequences of pictures, matrices and linear operators, and general sampled functions of several variables. The method selects a most efficient orthogonal representation from among a large number of possibilities. The efficiency functional need only be additive across direct sum decompositions. We present a description of the method for pictures, namely functions of two variables, using Shannon entropy as the efficiency functional, and mean-square deviation as the error criterion.

Best basis method. In Appendix II is developed a method for generating a library of orthogonal vectors in $\mathbf{R}^n$ (for large n) together with a notion of admissible subsets of these vectors. Admissible subsets form orthonormal bases of wavelet-packets, which because of their homogeneous tree structure may be rapidly searched for functional extrema. We can use a family of orthonormal vectors well suited to representing functions of 2 variables. These are products of quadrature mirror filters, as defined below:

Let $\{p_k\}, \{q_k\}$ belong to l^1, and define two decimating convolution operators $P : l^2 \to l^2$, $Q : l^2 \to l^2$ as follows:

$$Pf_k = \sum_{j=-\infty}^{\infty} p_j f_{j+2k}, \qquad Qf_k = \sum_{j=-\infty}^{\infty} q_j f_{j+2k}.$$

P and Q are called *quadrature mirror filters* if they satisfy an orthogonality condition:

$$PQ^* = QP^* = 0,$$

where P^* denotes the adjoint of P, and Q^* the adjoint of Q. They are further called *perfect reconstruction filters* if they satisfy the condition

$$P^*P + Q^*Q = I,$$

where I is the identity operator. These conditions translate to restrictions on the sequences $\{p_k\}, \{q_k\}$. Let m_0, m_1 be (bounded) functions defined by

$$m_0(\xi) = \sum_{k=-\infty}^{\infty} p_k e^{ik\xi}, \qquad m_1(\xi) = \sum_{k=-\infty}^{\infty} q_k e^{ik\xi}.$$

Then P, Q are quadrature mirror filters if and only if the matrix below is unitary for all ξ:

$$\begin{pmatrix} m_0(\xi) & m_0(\xi+\pi) \\ m_1(\xi) & m_1(\xi+\pi) \end{pmatrix}$$

This fact is proved in [D].

Now we can define a number of orthogonal 2-dimensional convolution-decimation filters in terms of P and Q. Four of them are simply tensor products of the pair of quadrature mirror filters, as in the construction of 2-dimensional wavelets of Meyer [M].

$$\begin{aligned}
F_0 &\stackrel{\text{def}}{=} P \otimes P, & F_0 v(x,y) &= \sum_{i,j} v(i,j) p_{2x+i} p_{2y+j} \\
F_1 &\stackrel{\text{def}}{=} P \otimes Q, & F_1 v(x,y) &= \sum_{i,j} v(i,j) p_{2x+i} q_{2y+j} \\
F_2 &\stackrel{\text{def}}{=} Q \otimes P, & F_2 v(x,y) &= \sum_{i,j} v(i,j) q_{2x+i} p_{2y+j} \\
F_3 &\stackrel{\text{def}}{=} Q \otimes Q, & F_3 v(x,y) &= \sum_{i,j} v(i,j) p_{2x+i} p2y + j
\end{aligned}$$

There are also pairs of extensions of one dimensional filters:

$$\begin{aligned}
P_Y &\stackrel{\text{def}}{=} I \otimes P, & P_Y v(x,y) &= \sum_{i,j} v(i,j) \delta_{x,i} p_{2y+j} = \sum_j v(x,j) p_{2y+j} \\
Q_Y &\stackrel{\text{def}}{=} I \otimes Q, & Q_Y v(x,y) &= \sum_{i,j} v(i,j) \delta_{x,i} q_{2y+j} = \sum_j v(x,j) q_{2y+j} \\
P_X &\stackrel{\text{def}}{=} P \otimes I, & P_X v(x,y) &= \sum_{i,j} v(i,j) p_{2x+i} \delta_{y,j} = \sum_i v(i,y) p_{2x+i} \\
Q_X &\stackrel{\text{def}}{=} Q \otimes I, & Q_X v(x,y) &= \sum_{i,j} v(i,j) q_{2x+i} \delta_{y,j} = \sum_i v(i,y) q_{2x+i}
\end{aligned}$$

These convolution-decimations have the following adjoints:

$$F_0^* v(x,y) = \sum_{i,j} v(i,j) p_{2i+x} p_{2j+y}$$
$$F_1^* v(x,y) = \sum_{i,j} v(i,j) p_{2i+x} q_{2j+y}$$
$$F_2^* v(x,y) = \sum_{i,j} v(i,j) q_{2i+x} p_{2j+y}$$
$$F_3^* v(x,y) = \sum_{i,j} v(i,j) p_{2i+x} p_{2j+y}$$

$$P_Y^* v(x,y) = \sum_{i,j} v(i,j) \delta_{x,i} p_{2j+y} = \sum_j v(x,j) p_{2j+y}$$
$$Q_Y^* v(x,y) = \sum_{i,j} v(i,j) \delta_{x,i} q_{2j+y} = \sum_j v(x,j) q_{2j+y}$$
$$P_X^* v(x,y) = \sum_{i,j} v(i,j) p_{2i+x} \delta_{y,j} = \sum_i v(i,y) p_{2i+x}$$
$$Q_X^* v(x,y) = \sum_{i,j} v(i,j) q_{2i+x} \delta_{y,j} = \sum_i v(i,y) q_{2i+x}$$

The orthogonality relations for this collection are as follows:

$$F_n F_m^* = \delta_{nm} I$$

$$I = F_0^* F_0 \oplus F_1^* F_1 \oplus F_2^* F_2 \oplus F_3^* F_3$$

$$P_X P_X^* = P_Y P_Y^* = Q_X Q_X^* = Q_Y Q_Y^* = I$$

$$P_X Q_X^* = Q_X P_X^* = P_Y Q_Y^* = Q_Y P_Y^* = 0$$

$$I = P_X^* P_X \oplus Q_X^* Q_X = P_Y^* P_Y \oplus Q_Y^* Q_Y$$

By a "picture" we will mean any function $S = S(x,y) \in l^2(\mathbf{Z}^2)$. The space $l^2(\mathbf{Z}^2)$ of pictures may be decomposed into a partially ordered set $\mathbf{W}$ of subspaces $W(n_X, n_Y, m_X. m_Y)$, where $m_X \geq 0$, $m_Y \geq 0$, $0 \leq n_X < 2^{m_X}$, and $0 \leq n_Y < 2^{m_Y}$. These are the images of orthogonal projections composed of products of convolution-

decimations. Put $W(0,0,0,0) = l^2$, and define recursively

$$W(2n_X + i, 2n_Y + j, m_X + 1, m_Y + 1) = F_{2i+j}^* F_{2i+j} W(n_X, n_Y, m_X, m_Y),$$

$$W(2n_X, n_Y, m_X + 1, m_Y) = P_X^* P_X W(n_X, n_Y, m_X, m_Y),$$

$$W(2n_X + 1, n_Y, m_X + 1, m_Y) = Q_X^* Q_X W(n_X, n_Y, m_X, m_Y),$$

$$W(n_X, 2n_Y, m_X, m_Y + 1) = P_Y^* P_Y W(n_X, n_Y, m_X, m_Y),$$

$$W(n_X, 2n_Y + 1, m_X, m_Y + 1) = Q_X^* Q_X W(n_X, n_Y, m_X, m_Y).$$

These subspaces may be partially ordered by a relation which we define recursively as well. We say W is a *precursor* of W' (write $W \preceq W'$) if they are equal or if $W' = G^*GW$ for a convolution-decimation G in the set $\{F_0, F_1, F_2, F_3, P_X, P_Y, Q_X, Q_Y\}$. We also say that $W \preceq W'$ if there is a finite sequence $V_1, \ldots, V_n$ of subspaces in $\mathbf{W}$ such that $W \preceq V_1 \preceq \cdots \preceq V_n \preceq W'$. This is well defined, since each application of G^*G increases at least one of the indices m_X or m_Y.

While $\{\mathbf{W}, \preceq\}$ is not a tree, it may be made into a tree if we select a subset of the relation $\preceq$. We will say that $W = W(n_X, n_Y, m_X, m_Y)$ is a *principal precursor* of $W' = W(n'_X, n'_Y, m'_X, m'_Y)$ (and write $W \prec W'$) if one of the following holds:

(1) $m_Y = m_X$, and $W' = G^*GW$ for $G \in \{F_0, F_1, F_2, F_3, P_X, P_Y, Q_X, Q_Y\}$, or

(2) $m_Y < m_X$, and $W' = G^*GW$ for $G \in \{P_X, Q_X\}$, or

(3) $m_Y > m_X$, and $W' = G^*GW$ for $G \in \{P_Y, Q_Y\}$.

Further, we will say that $W \prec W'$ if there is a finite sequence $V_0, \ldots, V_n$ of subspaces in $\mathbf{W}$ with $W \prec V_0 \prec \cdots \prec V_n \prec W'$. The relation $\prec$ is well defined, since it is a subrelation of $\preceq$, and it is not hard to see that every subspace $W \in \mathbf{W}$ has a unique first principal precursor. Therefore, $\{\mathbf{W}, \prec\}$ forms a (nonhomogeneous) tree, with $W(0,0,0,0)$ at its root.

Subspaces of a single principal precursor $W \in \mathbf{W}$ will be called its *children*. By the orthogonality condition,

$$\text{(F)} \qquad W = F_0^* F_0 W \oplus F_1^* F_1 W \oplus F_2^* F_2 W \oplus F_3^* F_3 W$$

$$\text{(X)} \qquad = P_X^* P_X W \oplus Q_X^* Q_X W$$

$$\text{(Y)} \qquad = P_Y^* P_Y W \oplus Q_Y^* Q_Y W.$$

The right hand side contains all the children of W, divided into the groups "F," "X," and "Y." Each labelled group of children provides a one-step orthogonal decomposition of W, and in general we will have three subsets of the children to choose from.

The coordinates with respect to the standard basis of $W(n_X, n_Y, m_X, m_Y)$ form the sequence $G_1 \ldots G_m W(0,0,0,0)$, where $m = \max\{m_X, m_Y\}$, and the particular filters $G_1 \ldots G_m$ are determined uniquely by n_X and n_Y. This is described in Appendix II, attached hereto. Therefore we can express in standard coordinates the orthogonal projections of $W(0,0,0,0)$ onto the complete tree of subspaces **W** by recursively convolving and decimating with the filters.

Relation with one-dimensional wave packets. Let w_{nmk} be a one-dimensional wave packet at sequency n, scale m and position k, in the notation of Appensix II. Then the element in the (k_X, k_Y) position of the subspace $W(n_X, n_Y, m_X, m_Y)$, at the index (x, y), may be written as $w_{n_X,m_X,k_X}(x)w_{n_Y,m_Y,k_Y}(y)$, which is evidently the tensor product of two one-dimensional wave packets. This is easily seen from the construction of $W(n_X, n_Y, m_X, m_Y)$: in the x-direction, there will be a total of m_X convolution-decimations in the order determined by n_X, with the result translated to position k_X, and similarly in the y-direction.

We will use the notation $w \otimes v$ for the tensor product of two one-dimensional wave packets, with the understanding that the second factor depends on the y-coordinate. Since the one-dimensional wave packets are themselves a redundant spanning set, their tensor products contain a redundancy of bases for $l^2\mathbf{R}^2$. We can search this collection of bases efficiently for a best-adapted basis, using any additive measure of information, in a manner only slightly more complicated than for the one dimensional case.

Selecting a best basis. Let $S = S(x, y)$ be a picture, and let **W** be a tree of wavelet packets. Choose an additive measure of information as described in Appendix II, and attribute to each node $W(n_X, n_Y, m_X, m_Y)$ the measure of information contained in the coordinates of S with respect to the wavelet packets it contains. For example, we may use Shannon entropy,

$$\mathcal{H}(W) = \sum_{k_X, k_Y} p^2 \log p^2,$$

where $p = \langle S, w_{n_X m_X k_X} \otimes w_{n_Y m_Y k_Y}\rangle$, and $W = W(n_X, m_X, n_Y, m_Y)$. We will choose an arbitrary maximum level in the tree **W**, and mark all of its nodes as "kept." Proceeding up from this level to the root, we will compare $\mathcal{H}(W)$ for a node W of the tree **W** to the minimum of $\sum_{W \prec W' \in F} \mathcal{H}(W')$, $\sum_{W \prec W' \in X} \mathcal{H}(W')$, and $\sum_{W \prec W' \in Y} \mathcal{H}(W')$. If $\mathcal{H}(W)$ is less, then mark W as "kept" and mark as "not kept" all nodes W' with $W \prec W'$; otherwise, mark$W$ as "not kept," but attribute to it the minimum of the entropies of its children. When this procedure terminates at the root, the nodes marked "kept" will comprise an orthogonal collection of wavelet packets.

It is not necessary to mark all descendants of a "kept" parent as not kept. The complexity of the search algorithm is $O(n \log n)$ if we never change the status of descendants, but instead take for the orthogonal collection only those nodes marked "kept" which have no ancestors marked "kept." These may be listed efficiently by indexing the tree in the preorder or depth-first order.

Error estimates for the best basis. Let $\mathcal{H}(S)$ denote the entropy of the picture S in the best basis found above. This quantity will be found attributed to node $W(0,0,0,0)$ at the end of the search. It is related to the classical Shannon entropy $\mathcal{H}_0$ by the equation

$$\mathcal{H}_0(S) = \|S\|^{-2}\mathcal{H}(S) + \log \|S\|^2$$

The largest $\exp \mathcal{H}_0(S) = \|S\|^2 \exp \mathcal{H}(S)/\|S\|^2$ terms of the wavelet packet expansion for S contain essentially all the energy of the original picture. Mean square error bounds for specific classes of signals are provided in Appendix IV.

REFERENCES

[D] Ingrid Daubechies, *Orthonormal bases of compactly supported wavelets*, Communications on Pure and Applied Mathematics **XLI** (1988), 909–996.

[M] Yves Meyer, *De la recherche petrolière à la geometrie des espaces de Banach en passant par les paraproduits*, Seminaire equations aux derivées partielles 1985-1986, École Polytechnique. Palaiseaux.

CLAIMS:

1. A method for encoding and decoding an input signal, comprising the steps of:

applying combinations of dilations and translations of a wavelet to the input signal to obtain processed values;

computing the information costs of the processed values;

selecting, as encoded signals, an orthogonal group of processed values, the selection being dependent on the computed information costs; and

decoding the encoded signals to obtain an output signal.

2. The method as defined by claim 1, wherein said wavelet has a plurality of vanishing moments.

3. The method as defined by claim 1 or 2, further comprising transmitting the encoded signals, and receiving the transmitted encoded signals before the decoding thereof.

4. The method as defined by claim 3, further comprising storing the encoded signals, and reading the stored encoded signals before the decoding thereof.

5. The method as defined by claim 2, wherein said step of applying combinations of dilations and translations of the wavelet to the input signal to obtain processed values comprises correlating said combinations of dilations and translations of the wavelet with the input signal.

6. The method as defined by claim 2 or 5, wherein combinations of dilations and translations of the wavelet are designated as wavelet-packets, and wherein the step of applying wavelet-packets to the input signal to obtain processed values includes: generating a tree of processed values, the tree having successive levels obtained by applying to the input signal, for a given level, wavelet-packets which are combinations of the wavelet-packets applied at a previous level.

7. The method as defined by claim 6, wherein the steps of computing information costs and selecting an orthogonal group of processed values includes performing said computing at a number of different levels of said tree, and performing said selecting

from among the different levels of the tree.

8. The method as defined by claim 2 or 7, wherein said step of selecting an orthogonal group of processed values comprises selecting an orthogonal group having a minimal information cost.

9. The method as defined by claim 8, wherein said step of selecting an orthogonal group of processed values includes generating encoded signals which represent said processed values in conjunction with their respective locations in said tree.

10. A method for encoding an input signal, comprising the steps of:

selecting a wavelet having a plurality of vanishing moments;

applying combinations of dilations and translations of the wavelet to the input signal to obtain processed values; and

selecting, as encoded signals, an orthogonal group of processed values.

11. The method as defined by claim 10, further comprising decoding the encoded signals.

12. The method as defined by claim 10 or 11, further comprising transmitting the encoded signals, and receiving the transmitted encoded signals before the decoding thereof.

13. The method as defined by claim 10, wherein said step of applying combinations of dilations and translations of the wavelet to the input signal to obtain processed values comprises correlating said combinations of dilations and translations of the wavelet with the input signal.

14. A method for encoding an input signal, comprising the steps of:

applying combinations of dilations and translations of a wavelet to the input signal to obtain processed values;

computing the information costs of the processed values; and

selecting, as encoded signals, an orthogonal group of processed values, the selection being dependent on the computed information costs.

15. The method as defined by claim 14, wherein said

wavelet has a plurality of vanishing moments.

16. The method as defined by claim 15, further comprising transmitting the encoded signals, and receiving the transmitted encoded signals before the decoding thereof.

17. The method as defined by claim 15, further comprising storing the encoded signals.

18. The method as defined by claim 15, wherein said step of applying combinations of dilations and translations of the wavelet to the input signal to obtain processed values comprises correlating said combinations of dilations and translations of the wavelet with the input signal.

19. The method as defined by claim 15 or 18, wherein combinations of dilations of the wavelet are designated as wavelet-packets, and wherein the step of applying wavelet-packets to the input signal to obtain processed values includes: generating a tree of processed values, the tree having successive levels obtained by applying to the input signal, for a given level, wavelet-packets which are combinations of the wavelet-packets applied at a previous level.

20. The method as defined by claim 19, wherein the steps of computing information costs and selecting an orthogonal group of processed values includes performing said computing at a number of different levels of said tree, and performing said selecting from among the different levels of the tree.

21. The method as defined by claim 15, wherein said step of selecting an orthogonal group of processed values comprises selecting an orthogonal group having a minimal information cost.

22. The method as defined by claim 20, wherein said step of selecting an orthogonal group of processed values comprises selecting an orthogonal group having a minimal information cost.

23. The method as defined by claim 22, wherein said step of selecting an orthogonal group of processed values includes generating encoded signals which represent said processed values in conjunction with their respective locations in said tree.

24. For use in a system which receives an encoded signal obtained by: applying combinations of dilations and translations of a wavelet having a plurality of vanishing moments to the input

signal to obtain processed values; and selecting, as encoded signals, an orthogonal group of processed values; a decoding method comprising: sequentially applying combinations of dilations and translations of a wavelet having a plurality of vanishing moments to the encoded signals to decode said encoded signals; and outputting the decoded result.

25. The decoding method as defined by claim 24, wherein the encoded signals include identification of the selected orthogonal group of processed values, and further comprising the step of determining said identification, and performing the sequential decoding procedure in accordance with said identification.

26. Apparatus for encoding an input signal, comprising:

means for applying combinations of dilations and translations of the wavelet to the input signal to obtain processed values;

means for computing the information costs of the processed values;

means for selecting, as encoded signals, an orthogonal group of processed values, the selection being dependent on the computed information costs.

27. Apparatus as defined by claim 26, wherein said wavelet has a plurality of vanishing moments.

28. Apparatus as defined by claim 27, wherein said means for applying combinations of dilations and translations of the wavelet to the input signal to obtain processed values comprises means for correlating said combinations of dilations and translations of the wavelet with the input signal.

29. Apparatus for encoding an input signal, comprising:

means for applying combinations of dilations and translations of a wavelet having a plurality of vanishing moments to the input signal to obtain processed values; and

means for selecting, as encoded signals, an orthogonal group of processed values.

30. For use in a system which receives an encoded signal obtained by: applying combinations of dilations and translations of a wavelet having a plurality of vanishing moments to the input signal to obtain processed values; and selecting, as encoded

signals, an orthogonal group of processed values; a decoding apparatus comprising: means for sequentially applying combinations of dilations and translations of a wavelet having a plurality of vanishing moments to the encoded signals to decode said encoded signals; and means for outputting the decoded result.

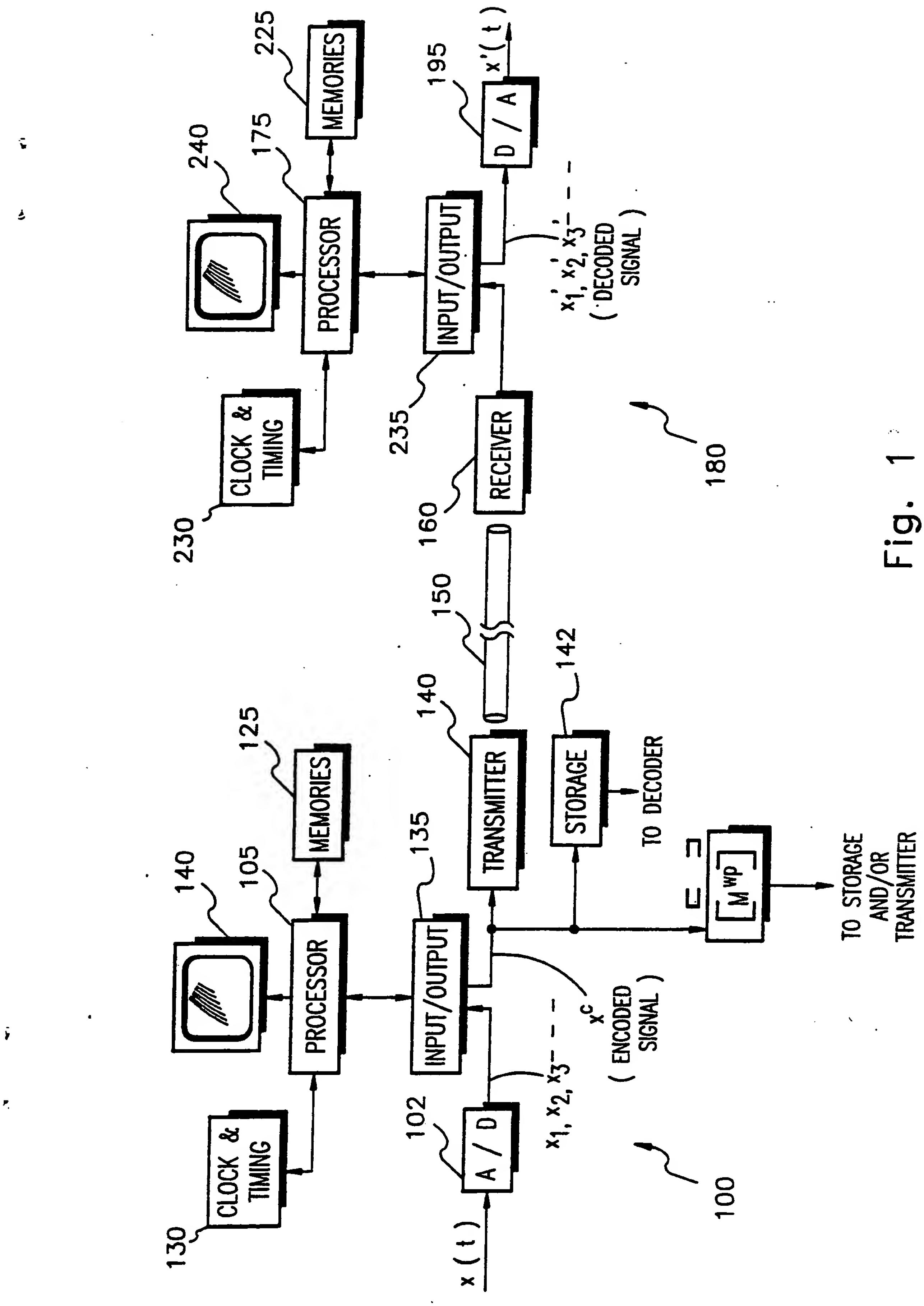

x (t)
A / D
102
x_1, x_2, x_3 - - -
100
CLOCK & TIMING
130
PROCESSOR
105
140
MEMORIES
125
INPUT/OUTPUT
135
x^c
(ENCODED SIGNAL)
TRANSMITTER
140
STORAGE
142
TO DECODER
$[M^{WP}]$
TO STORAGE AND/OR TRANSMITTER
150
160
RECEIVER
180
235
INPUT/OUTPUT
CLOCK & TIMING
230
PROCESSOR
175
240
MEMORIES
225
x_1', x_2', x_3' - - -
(DECODED SIGNAL)
D / A
195
x'(t)

Fig. 1

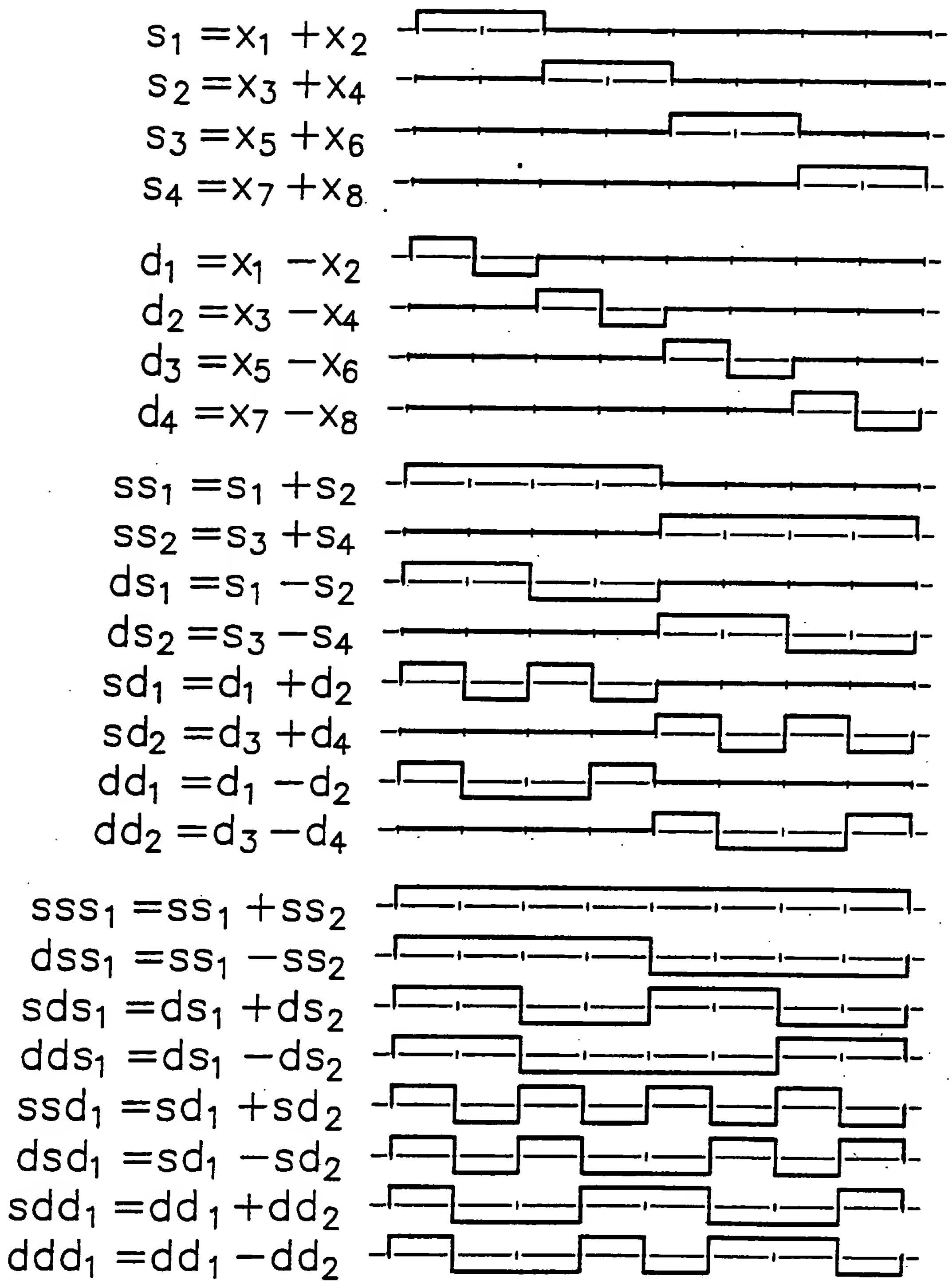

$s_1 = x_1 + x_2$
$s_2 = x_3 + x_4$
$s_3 = x_5 + x_6$
$s_4 = x_7 + x_8$
$d_1 = x_1 - x_2$
$d_2 = x_3 - x_4$
$d_3 = x_5 - x_6$
$d_4 = x_7 - x_8$
$ss_1 = s_1 + s_2$
$ss_2 = s_3 + s_4$
$ds_1 = s_1 - s_2$
$ds_2 = s_3 - s_4$
$sd_1 = d_1 + d_2$
$sd_2 = d_3 + d_4$
$dd_1 = d_1 - d_2$
$dd_2 = d_3 - d_4$
$sss_1 = ss_1 + ss_2$
$dss_1 = ss_1 - ss_2$
$sds_1 = ds_1 + ds_2$
$dds_1 = ds_1 - ds_2$
$ssd_1 = sd_1 + sd_2$
$dsd_1 = sd_1 - sd_2$
$sdd_1 = dd_1 + dd_2$
$ddd_1 = dd_1 - dd_2$

Fig. 2

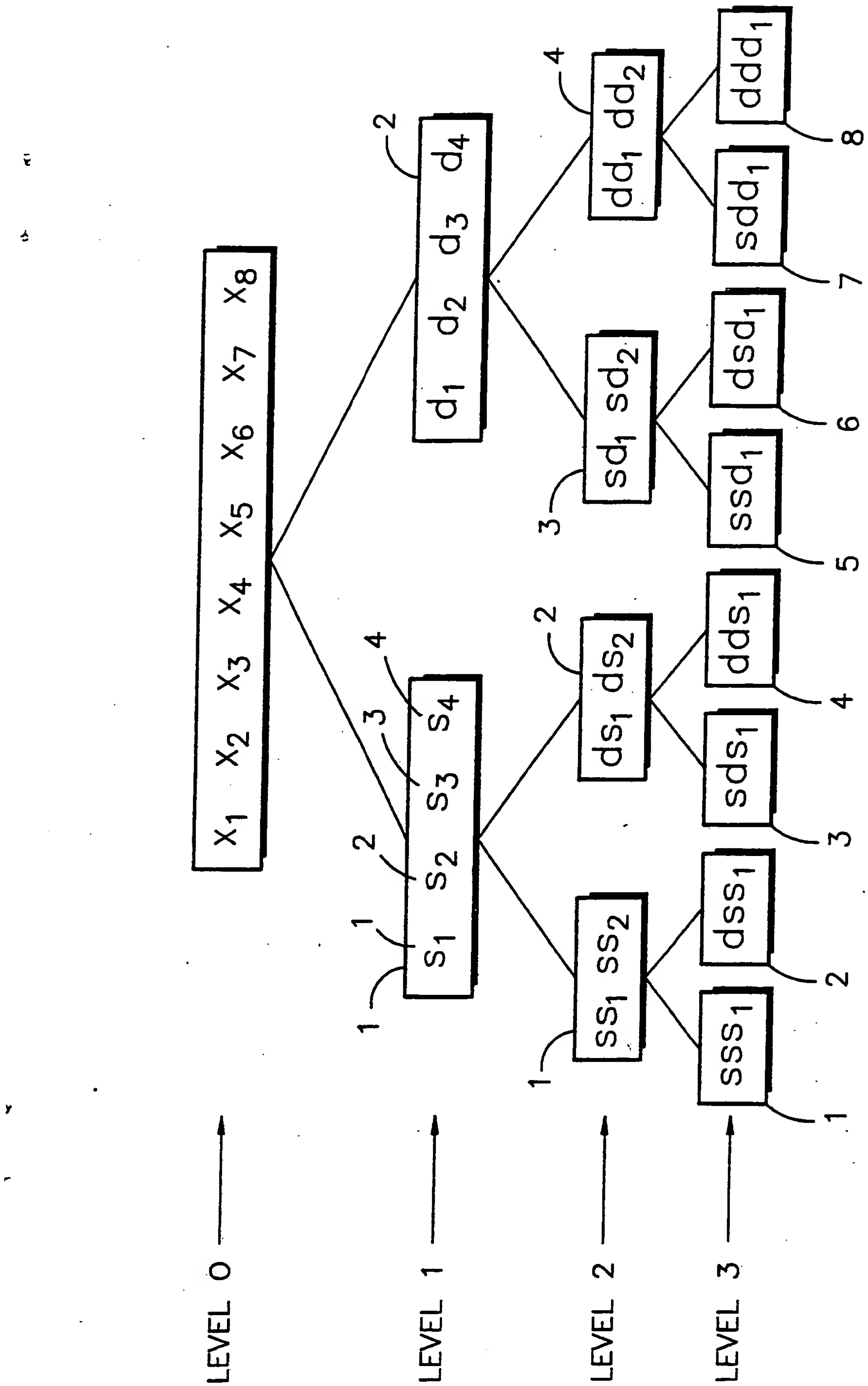

LEVEL 0
LEVEL 1
LEVEL 2
LEVEL 3
X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8
s_1 s_2 s_3 s_4
d_1 d_2 d_3 d_4
ss_1 ss_2
ds_1 ds_2
sd_1 sd_2
dd_1 dd_2
sss_1
dss_1
sds_1
dds_1
ssd_1
dsd_1
sdd_1
ddd_1
1
2
3
4
5
6
7
8

Fig. 3

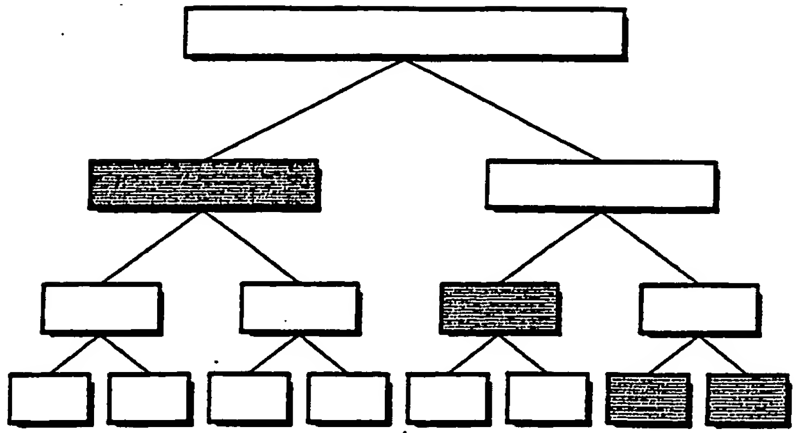


Fig. 4A

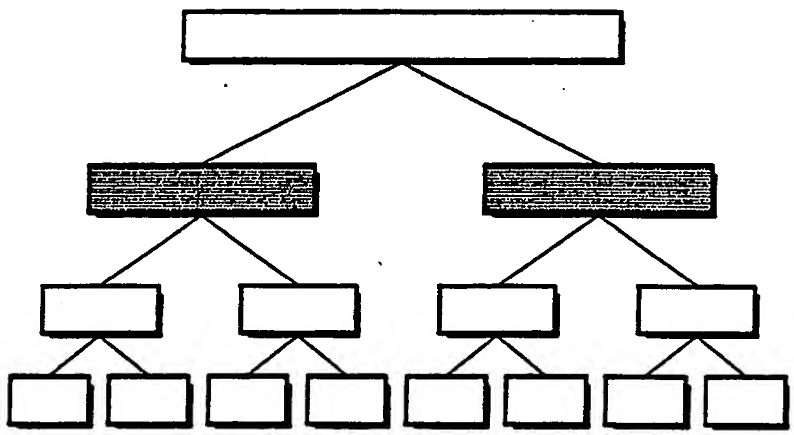


Fig. 4B

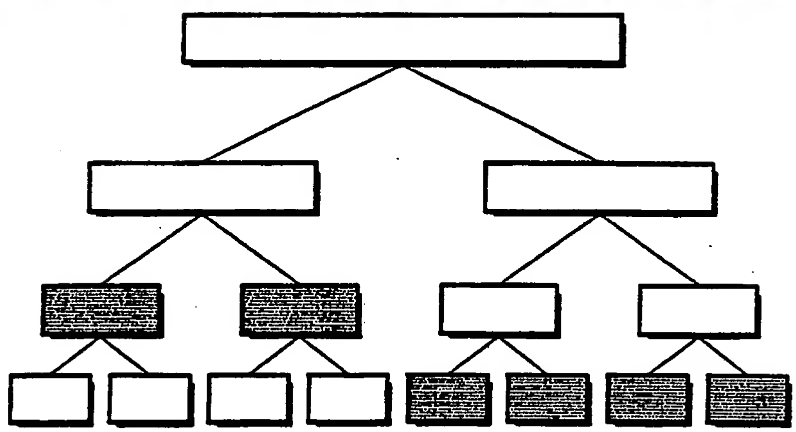


Fig. 4C

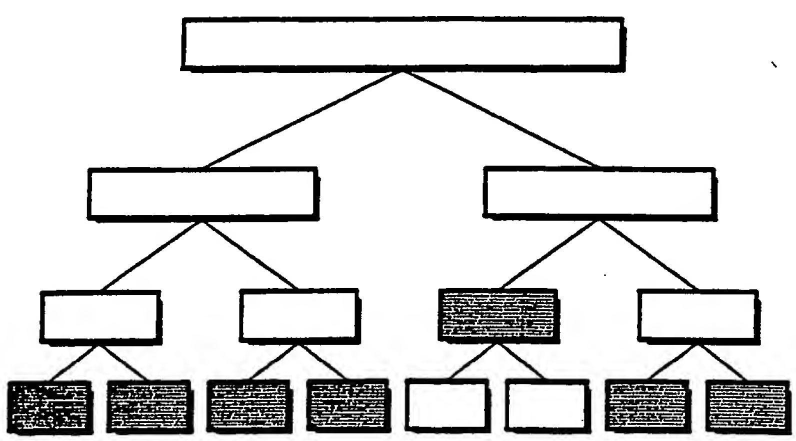


Fig. 4D

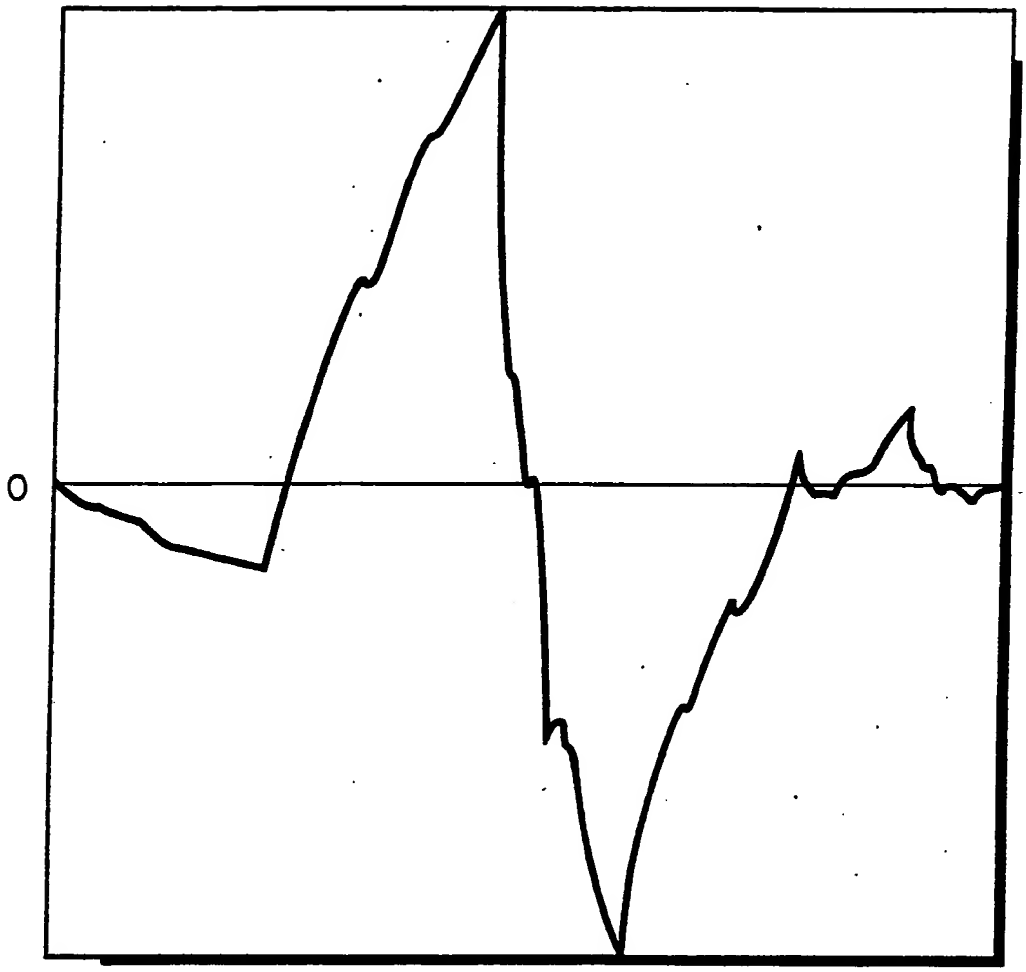


Fig. 5

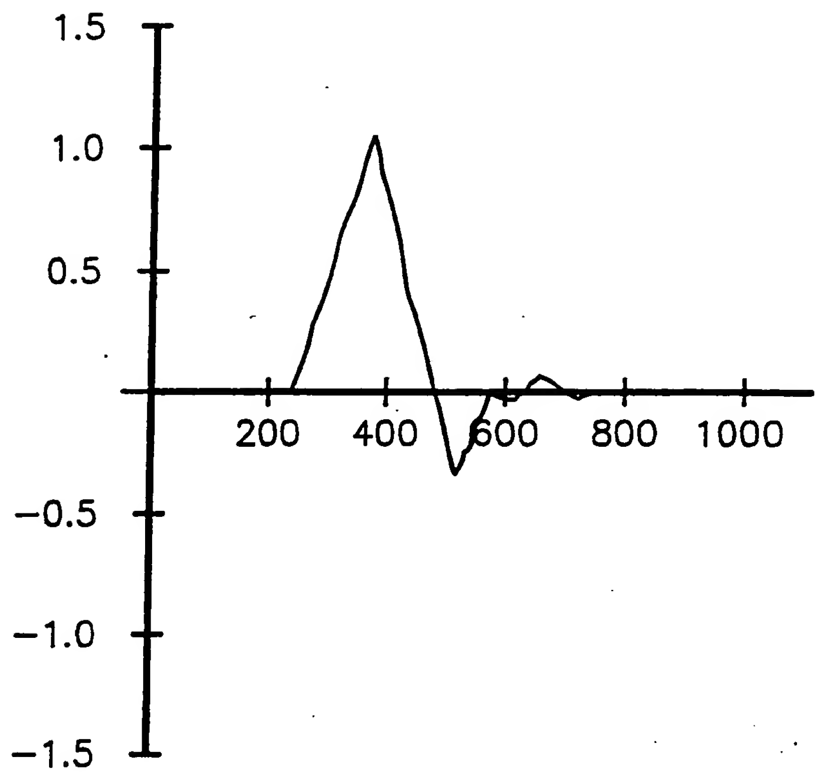

1.5
1.0
0.5
-0.5
-1.0
-1.5
200
400
600
800
1000

Fig. 6

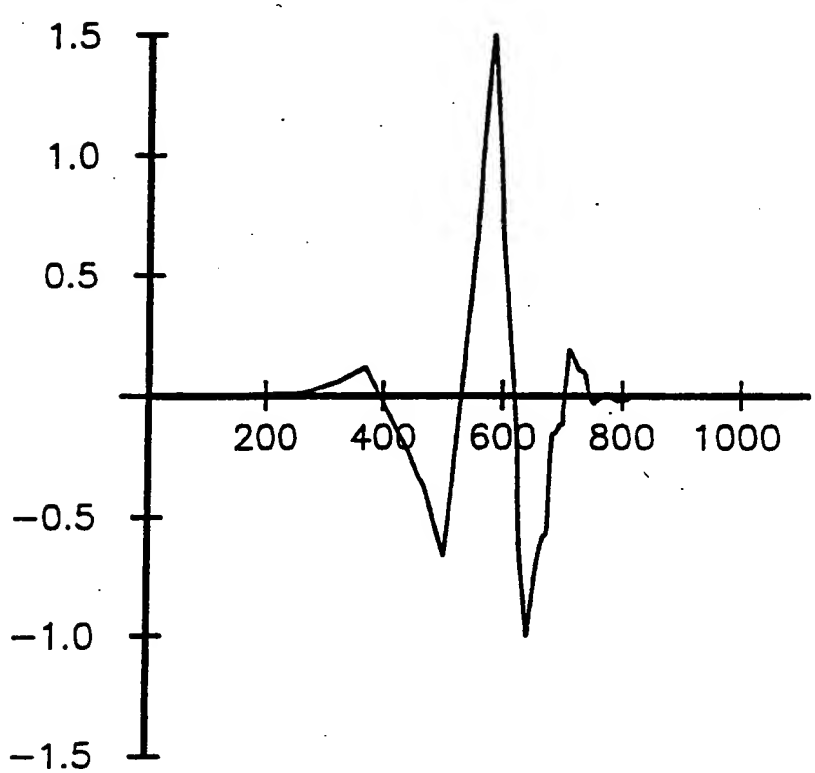

1.5
1.0
0.5
-0.5
-1.0
-1.5
200
400
600
800
1000

Fig. 7

Fig. 8

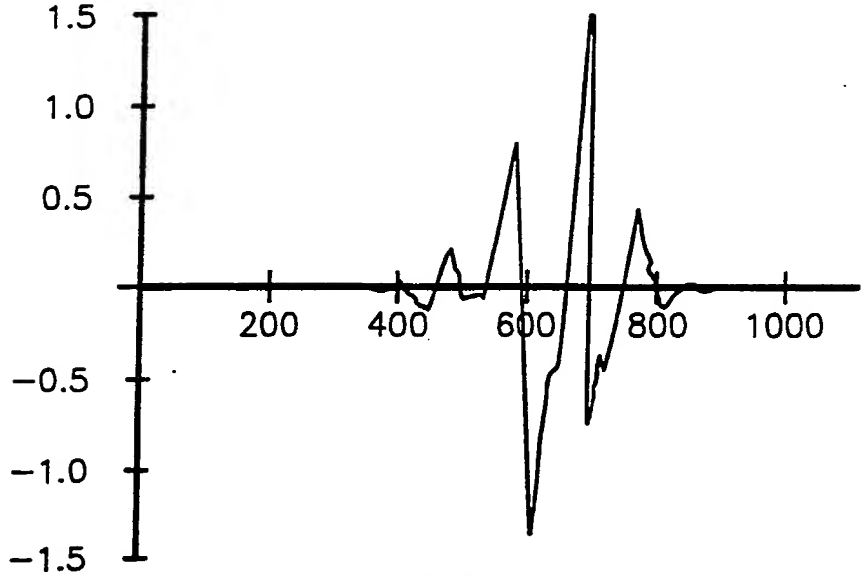


Fig. 9

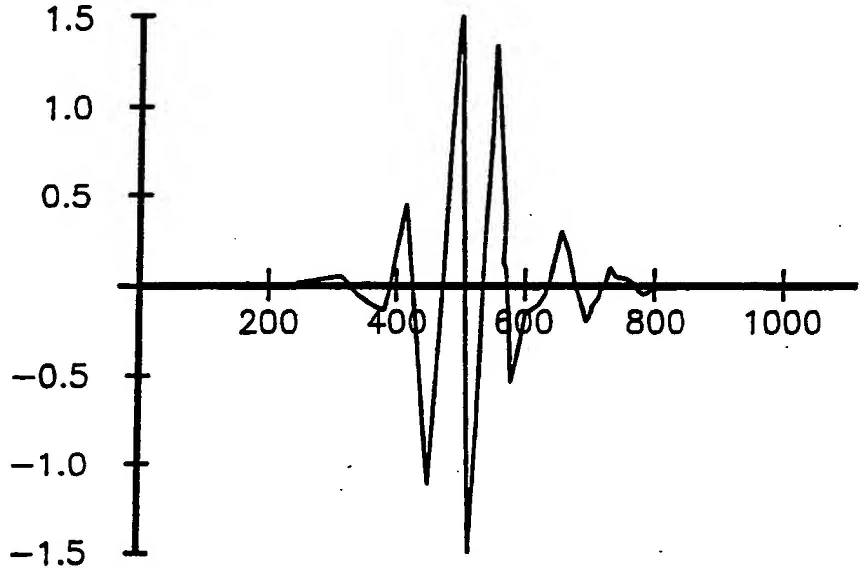
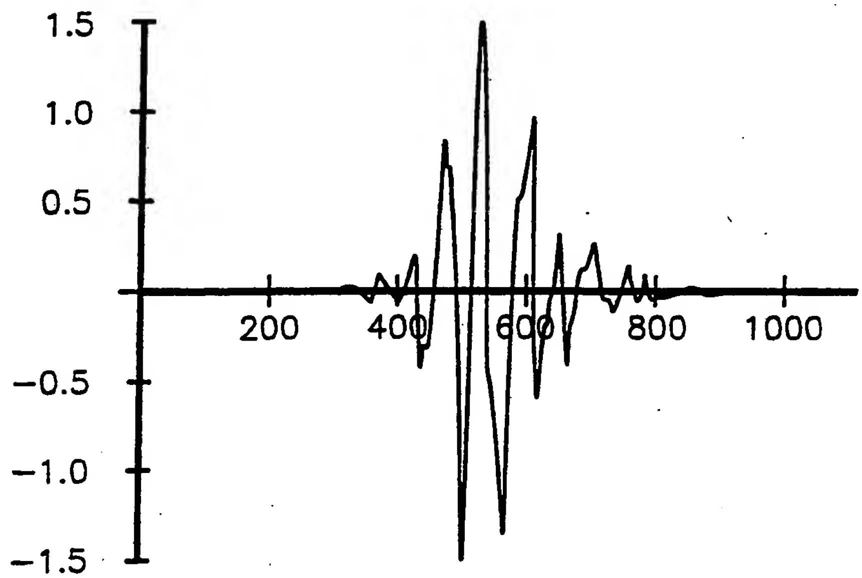


Fig. 10



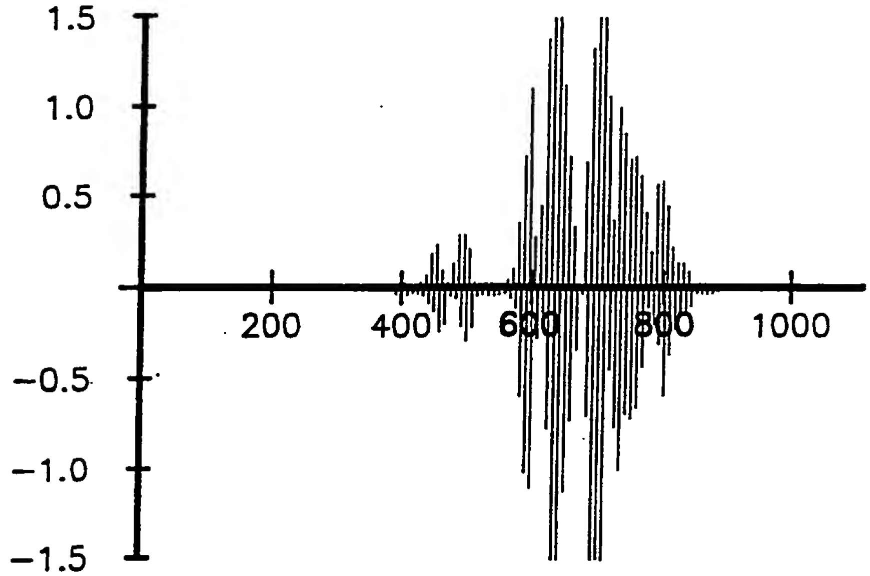


Fig. 11

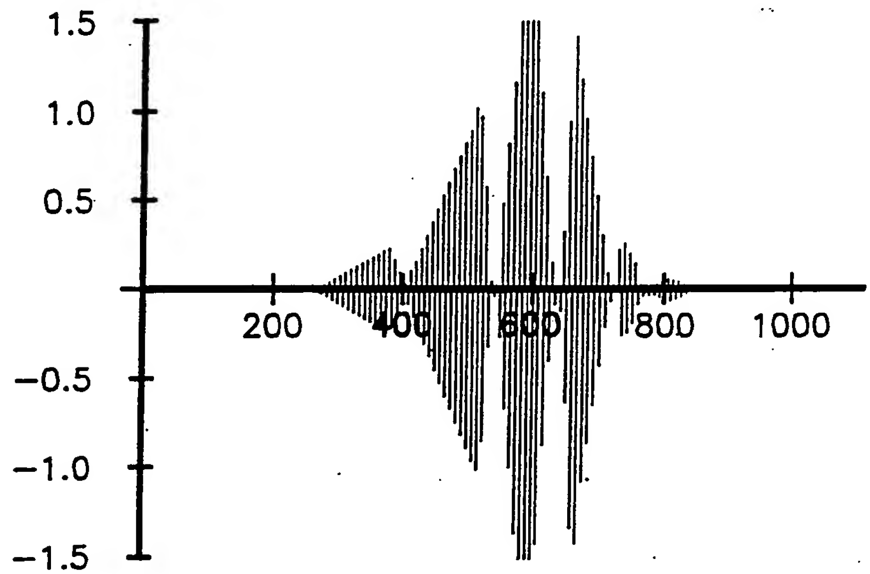


Fig. 12

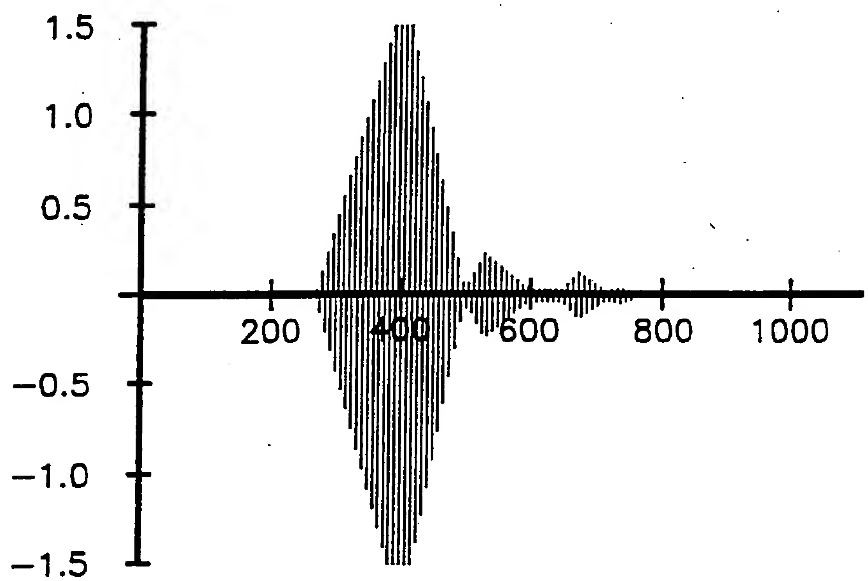


Fig. 13

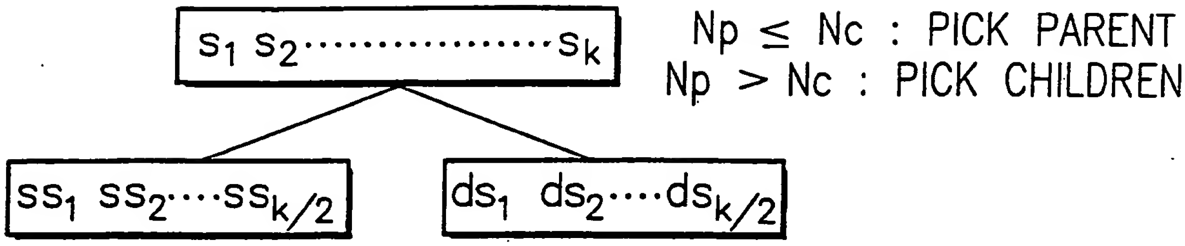


Fig. 14

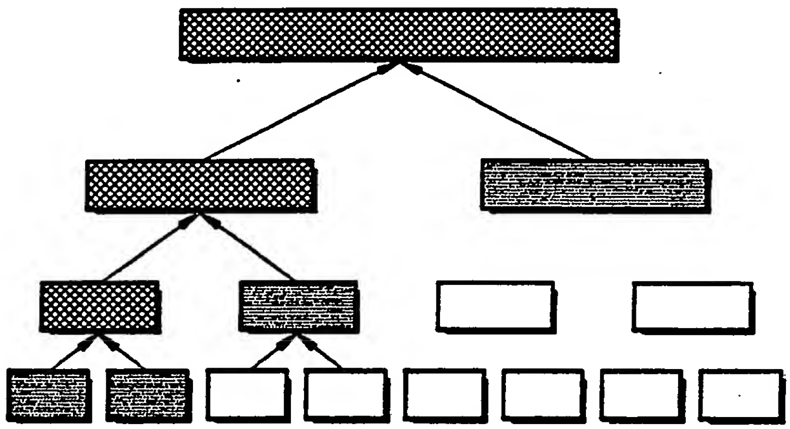


Fig. 15A

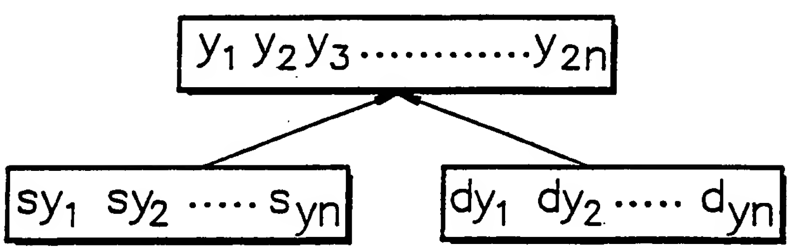


FiG. 15B

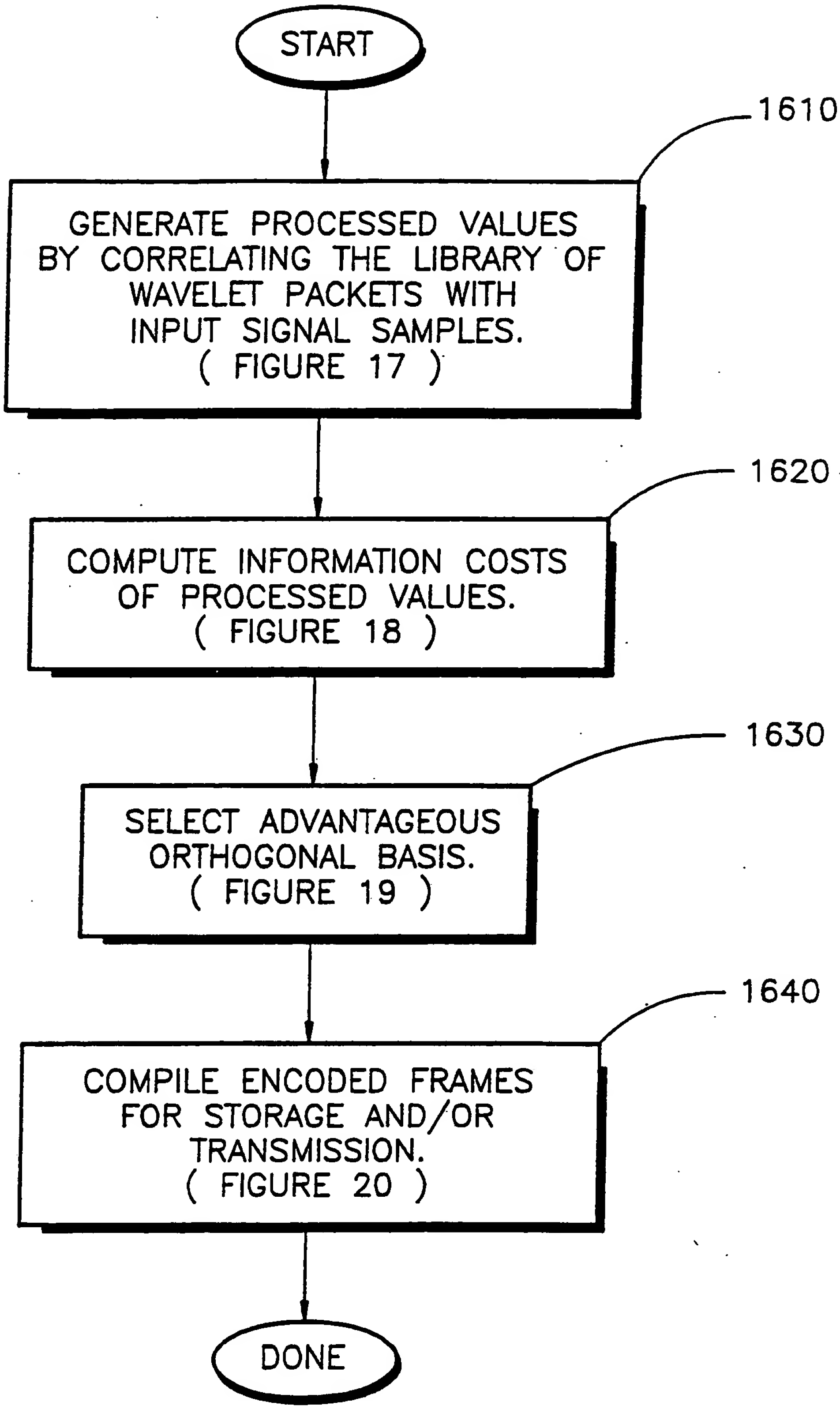

START
1610
GENERATE PROCESSED VALUES BY CORRELATING THE LIBRARY OF WAVELET PACKETS WITH INPUT SIGNAL SAMPLES. (FIGURE 17)
1620
COMPUTE INFORMATION COSTS OF PROCESSED VALUES. (FIGURE 18)
1630
SELECT ADVANTAGEOUS ORTHOGONAL BASIS. (FIGURE 19)
1640
COMPILE ENCODED FRAMES FOR STORAGE AND/OR TRANSMISSION. (FIGURE 20)
DONE

Fig. 16

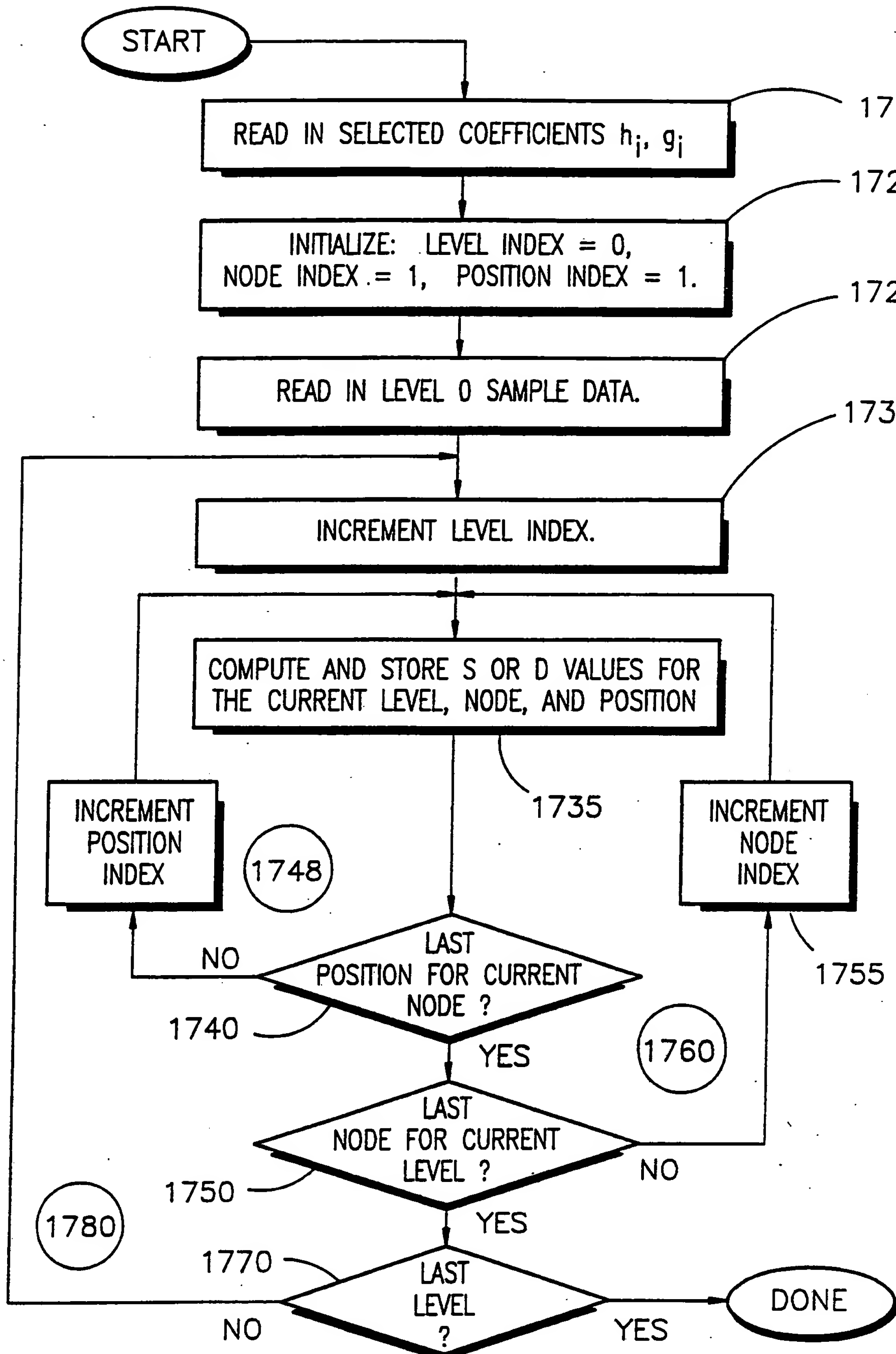

START
READ IN SELECTED COEFFICIENTS h_i, g_i
1710
INITIALIZE: LEVEL INDEX = 0,
NODE INDEX = 1, POSITION INDEX = 1.
1720
READ IN LEVEL 0 SAMPLE DATA.
1725
INCREMENT LEVEL INDEX.
1730
COMPUTE AND STORE S OR D VALUES FOR
THE CURRENT LEVEL, NODE, AND POSITION
1735
INCREMENT POSITION INDEX
1748
INCREMENT NODE INDEX
1755
LAST POSITION FOR CURRENT NODE ?
NO
YES
1740
1760
LAST NODE FOR CURRENT LEVEL ?
NO
YES
1750
1780
1770
LAST LEVEL ?
NO
YES
DONE

Fig. 17

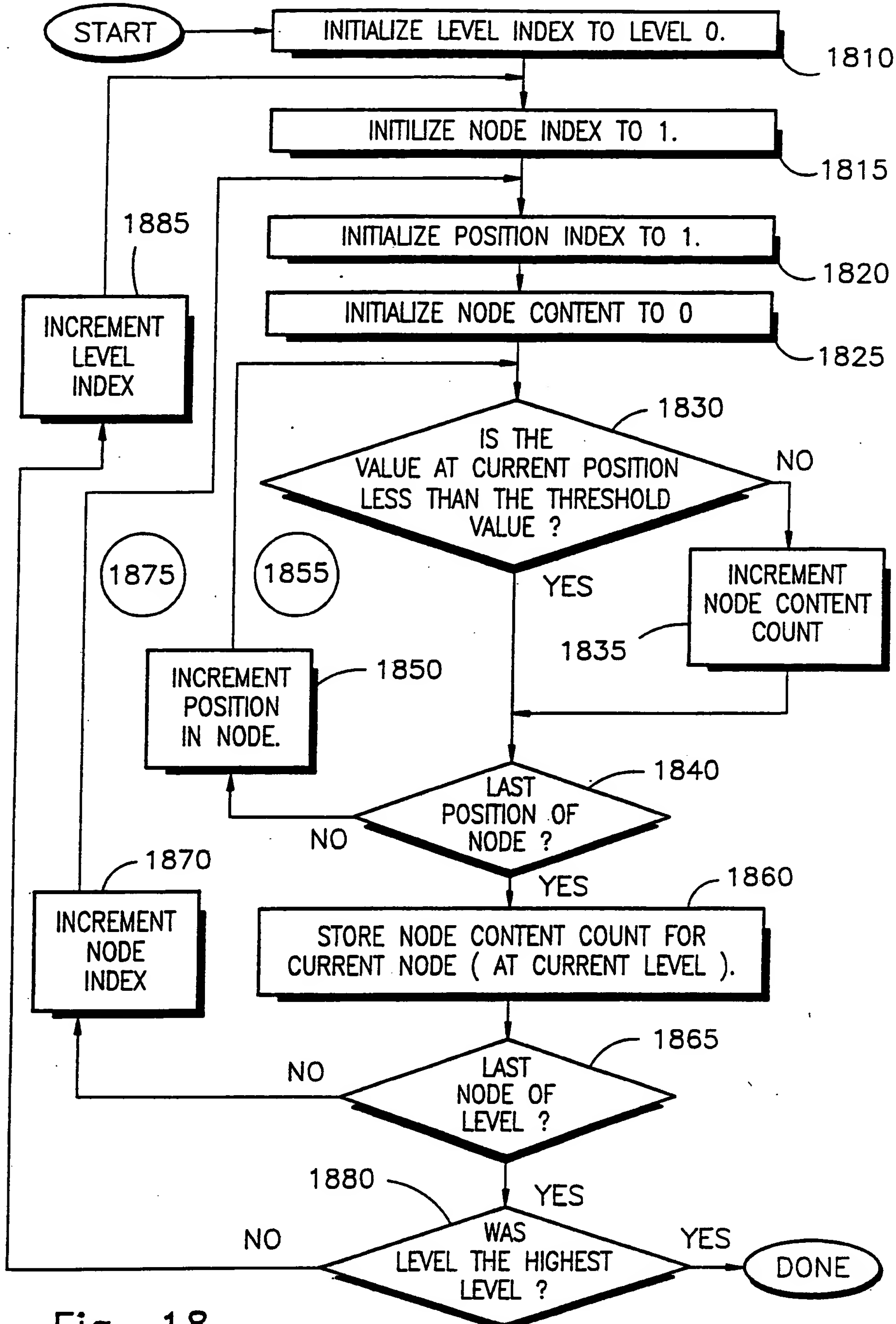

START
INITIALIZE LEVEL INDEX TO LEVEL 0.
1810
INITILIZE NODE INDEX TO 1.
1815
INITIALIZE POSITION INDEX TO 1.
1820
INITIALIZE NODE CONTENT TO 0
1825
1885
INCREMENT LEVEL INDEX
1830
IS THE VALUE AT CURRENT POSITION LESS THAN THE THRESHOLD VALUE ?
NO
YES
INCREMENT NODE CONTENT COUNT
1835
1875
1855
INCREMENT POSITION IN NODE.
1850
1840
LAST POSITION OF NODE ?
NO
YES
1870
INCREMENT NODE INDEX
1860
STORE NODE CONTENT COUNT FOR CURRENT NODE (AT CURRENT LEVEL).
1865
LAST NODE OF LEVEL ?
NO
YES
1880
WAS LEVEL THE HIGHEST LEVEL ?
NO
YES
DONE

Fig. 18

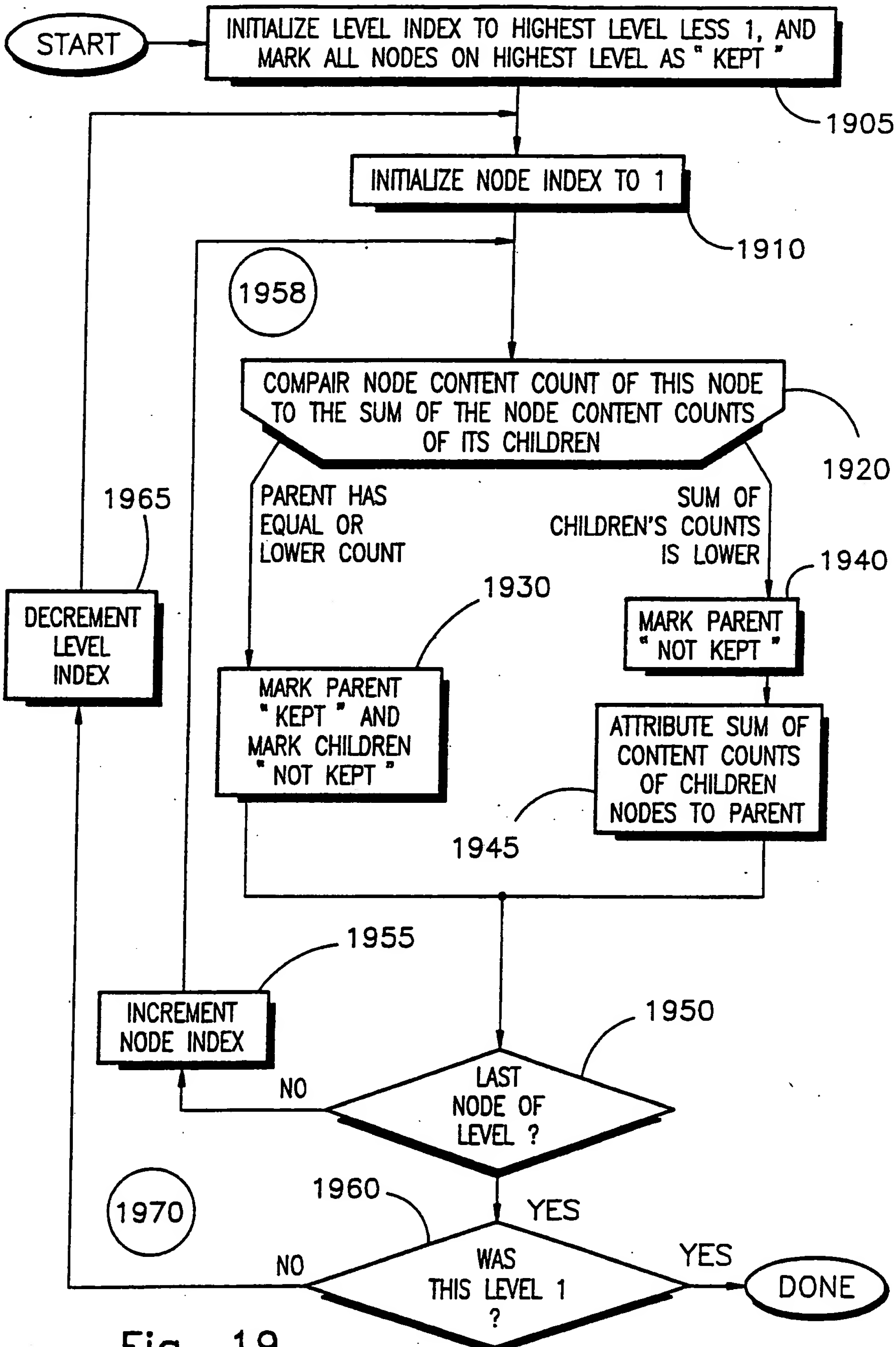

START
INITIALIZE LEVEL INDEX TO HIGHEST LEVEL LESS 1, AND MARK ALL NODES ON HIGHEST LEVEL AS " KEPT "
1905
INITIALIZE NODE INDEX TO 1
1910
1958
COMPAIR NODE CONTENT COUNT OF THIS NODE TO THE SUM OF THE NODE CONTENT COUNTS OF ITS CHILDREN
1920
PARENT HAS EQUAL OR LOWER COUNT
SUM OF CHILDREN'S COUNTS IS LOWER
1965
DECREMENT LEVEL INDEX
1930
1940
MARK PARENT " NOT KEPT "
MARK PARENT " KEPT " AND MARK CHILDREN " NOT KEPT "
ATTRIBUTE SUM OF CONTENT COUNTS OF CHILDREN NODES TO PARENT
1945
1955
INCREMENT NODE INDEX
1950
NO
LAST NODE OF LEVEL ?
1960
YES
1970
NO
WAS THIS LEVEL 1 ?
YES
DONE

Fig. 19

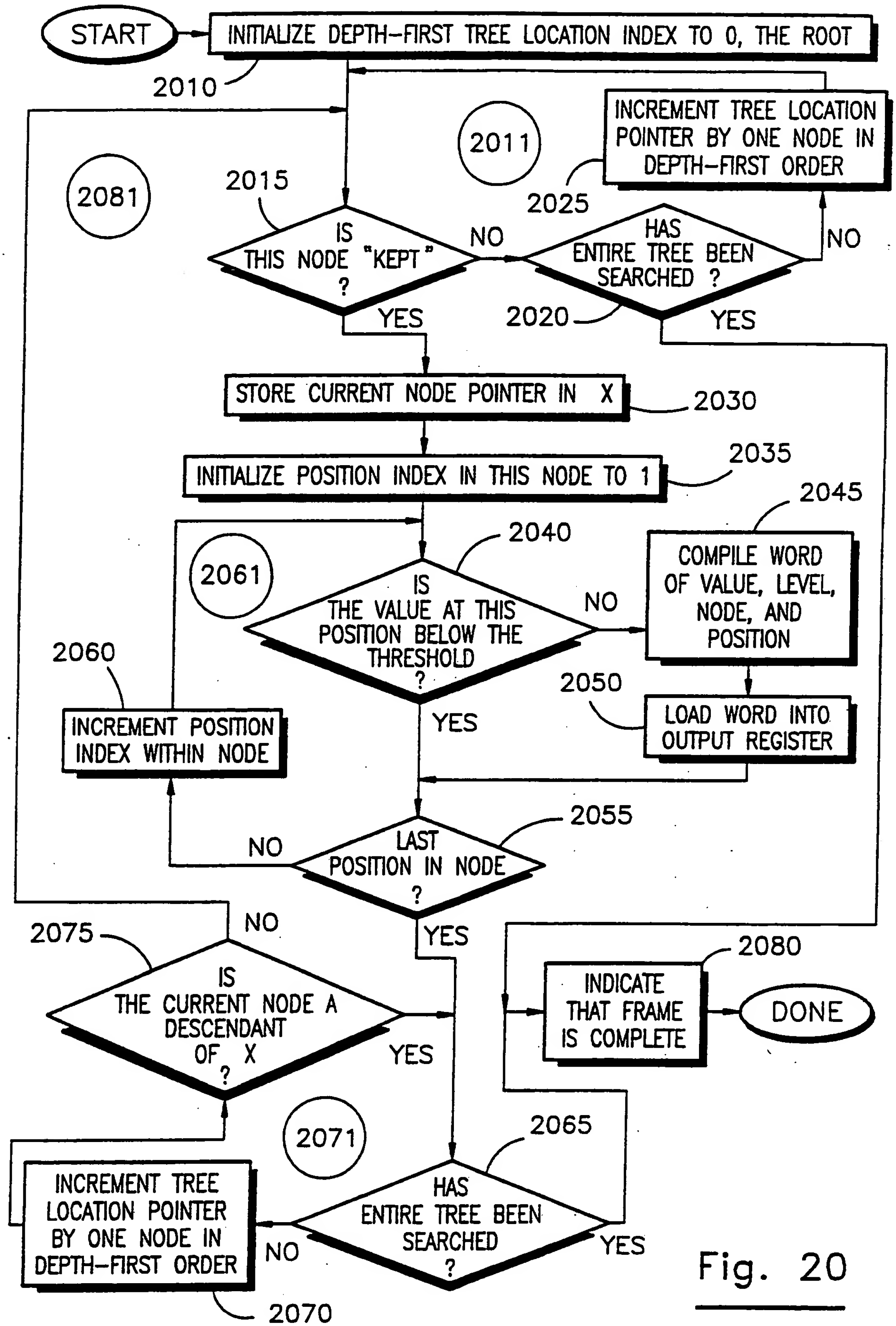

START
INITIALIZE DEPTH-FIRST TREE LOCATION INDEX TO 0, THE ROOT
2010
2011
INCREMENT TREE LOCATION POINTER BY ONE NODE IN DEPTH-FIRST ORDER
2081
2015
2025
IS THIS NODE "KEPT" ?
NO
HAS ENTIRE TREE BEEN SEARCHED ?
NO
YES
2020
YES
STORE CURRENT NODE POINTER IN X
2030
2035
INITIALIZE POSITION INDEX IN THIS NODE TO 1
2045
2040
2061
IS THE VALUE AT THIS POSITION BELOW THE THRESHOLD ?
NO
COMPILE WORD OF VALUE, LEVEL, NODE, AND POSITION
2060
2050
YES
LOAD WORD INTO OUTPUT REGISTER
INCREMENT POSITION INDEX WITHIN NODE
2055
NO
LAST POSITION IN NODE ?
2075
NO
YES
2080
IS THE CURRENT NODE A DESCENDANT OF X ?
INDICATE THAT FRAME IS COMPLETE
DONE
YES
2071
2065
INCREMENT TREE LOCATION POINTER BY ONE NODE IN DEPTH-FIRST ORDER
NO
HAS ENTIRE TREE BEEN SEARCHED ?
YES
2070

Fig. 20

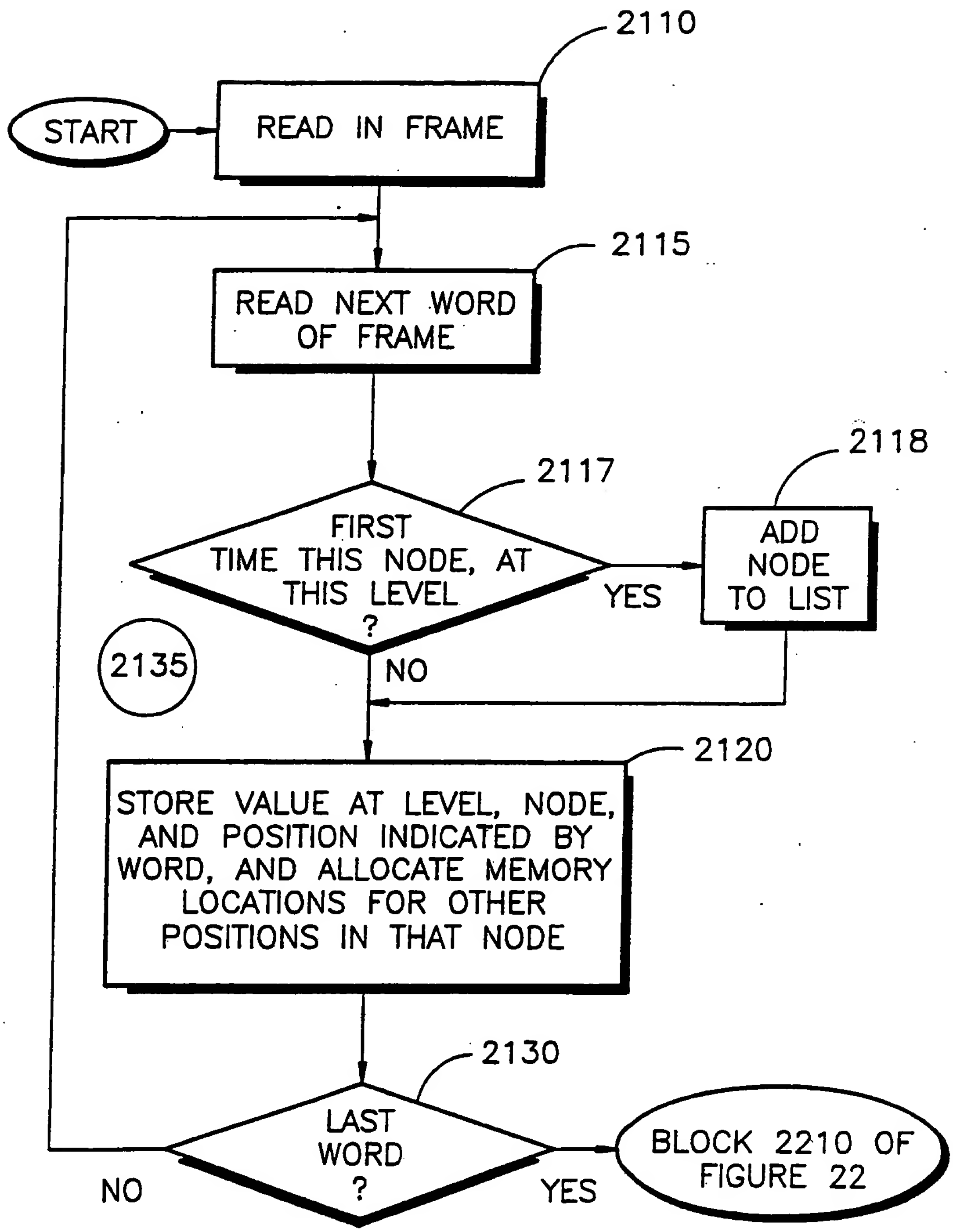

2110
START
READ IN FRAME
2115
READ NEXT WORD OF FRAME
2117
FIRST TIME THIS NODE, AT THIS LEVEL ?
YES
2118
ADD NODE TO LIST
NO
2135
2120
STORE VALUE AT LEVEL, NODE, AND POSITION INDICATED BY WORD, AND ALLOCATE MEMORY LOCATIONS FOR OTHER POSITIONS IN THAT NODE
2130
LAST WORD ?
NO
YES
BLOCK 2210 OF FIGURE 22

Fig. 21

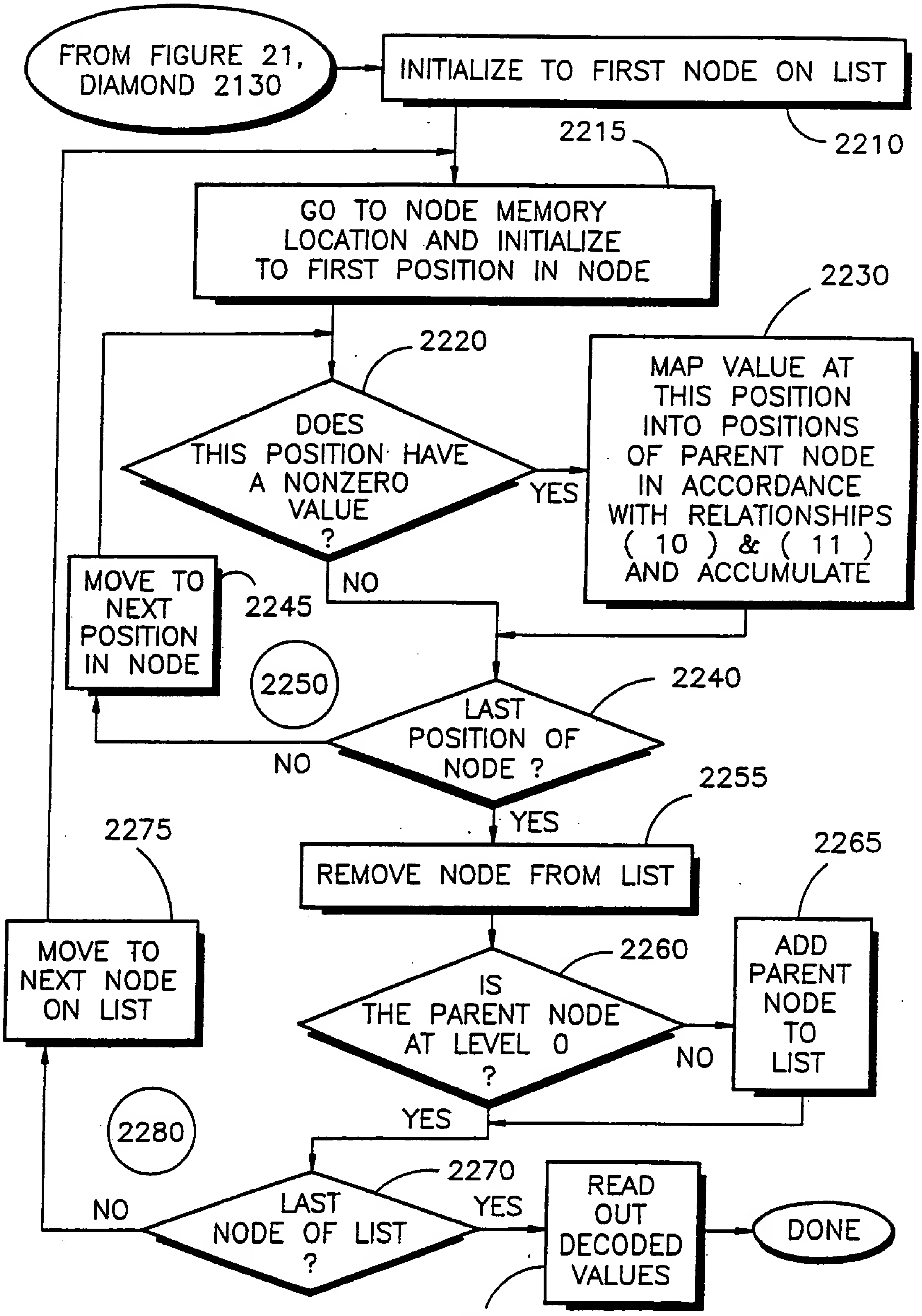

FROM FIGURE 21, DIAMOND 2130
INITIALIZE TO FIRST NODE ON LIST
2210
2215
GO TO NODE MEMORY LOCATION AND INITIALIZE TO FIRST POSITION IN NODE
2230
2220
DOES THIS POSITION HAVE A NONZERO VALUE ?
YES
MAP VALUE AT THIS POSITION INTO POSITIONS OF PARENT NODE IN ACCORDANCE WITH RELATIONSHIPS (10) & (11) AND ACCUMULATE
NO
MOVE TO NEXT POSITION IN NODE
2245
2250
2240
LAST POSITION OF NODE ?
NO
YES
2255
REMOVE NODE FROM LIST
2275
2265
MOVE TO NEXT NODE ON LIST
2260
IS THE PARENT NODE AT LEVEL 0 ?
NO
ADD PARENT NODE TO LIST
YES
2280
2270
LAST NODE OF LIST ?
NO
YES
READ OUT DECODED VALUES
DONE
2290

Fig. 22

INTERNATIONAL SEARCH REPORT

International Application No PCT/US91/03504

I. CLASSIFICATION OF SUBJECT MATTER (if several classification symbols apply, indicate all) [3]

According to International Patent Classification (IPC) or to both National Classification and IPC

IPC(5): G06G 7/00
US CL: 364/807

II. FIELDS SEARCHED

Minimum Documentation Searched [4]

Classification System	Classification Symbols
US	364/807,826,715.1,724.14,724.12,725,728.01,728.03,421 358/261.3,262.1,432,426 73/625,628 367/38,59 128/660.01

Documentation Searched other than Minimum Documentation to the Extent that such Documents are Included in the Fields Searched [5]

III. DOCUMENTS CONSIDERED TO BE RELEVANT [14]

Category *	Citation of Document, [16] with indication, where appropriate, of the relevant passages [17]	Relevant to Claim No. [18]
A	US, A, 4,706,499 (ANDERSON), 17 NOV. 1987)	1-30
A	US, A, 4,922,465 (PIEPRZAK ET AL) 1 MAY 1990	1-30
A,P	US, A, 5,000,183 (BONNEFOUS) 19 MAR 1991	1-30
Y,P	US, A, 5,014,134 (LAWTON ET AL) 7 MAY 1991 See Figure 1	1-30
Y,P	US, A, 4,974,187 (LAWTON) 27 NOV. 1990 See Abstract and Fig. 12	1-30
Y	I. Darbechies, "Oxthonormul Bases of Compactly Supported Wavelets", Comm. Pure, Applied Math, XL1 1988. See whole document.	1-30

* Special categories of cited documents: [15]

"A" document defining the general state of the art which is not considered to be of particular relevance

"E" earlier document but published on or after the international filing date

"L" document which may throw doubts on priority claim(s) or which is cited to establish the publication date of another citation or other special reason (as specified)

"O" document referring to an oral disclosure, use, exhibition or other means

"P" document published prior to the international filing date but later than the priority date claimed

"T" later document published after the international filing date or priority date and not in conflict with the application but cited to understand the principle or theory underlying the invention

"X" document of particular relevance; the claimed invention cannot be considered novel or cannot be considered to involve an inventive step

"Y" document of particular relevance; the claimed invention cannot be considered to involve an inventive step when the document is combined with one or more other such documents, such combination being obvious to a person skilled in the art.

"&" document member of the same patent family

IV. CERTIFICATION

Date of the Actual Completion of the International Search [2]	Date of Mailing of this International Search Report [2]
27 Aug. 1991	01 OCT 1991
International Searching Authority [1]	**Signature of Authorized Officer** [10]
ISA/US	NGUYEN NGOC-HO INTERNATIONAL DIVISION Jim Trammell Ngoc-Ho Nguyen

International Application No. PCT/US91/03504

III. DOCUMENTS CONSIDERED TO BE RELEVANT (CONTINUED FROM THE SECOND SHEET)		
Category *	Citation of Document, [16] with indication, where appropriate, of the relevant passages [17]	Relevant to Claim No [18]
Y	S. Mallat, "Review of Multifrequency Channel Decomposition of Images and Wavelet Models" Technical Report 412, Robotics Report 1/8, NYU(1988) See whole document.	1-30
Y	T. Meyer, Wavelets and Operators, Analysis at Urbana, volil, edited by E. Berkson,N.T. Peck and J. Uhl, Lordon Math, Society, Lecture Notes Series 137, 1989 See whole document.	1-30
Y	S.G. Mallat, "A Theory For Multiresoluction Signal Decomposition": The Wavelet Representation", IEEE Transactions on Pattern Analysis and Machine Intelligence.Vor.II, No.7, July 1989 See whole document.	1-30
Y	G. Strans, "Wavelets and Dilation Equations: A Brief Introduction", SIAM Review, August, 1989 See whole document.	1-30
Y,P	R.R. Loifman, "Wavelet Analysis and Signal Processing", IMA Volumes In Mathematics and Its Applications, Vol. 22, Springer Verlag, 1990 See whole document	1-30

International Application No. PCT/US91/03504

FURTHER INFORMATION CONTINUED FROM THE SECOND SHEET

V. [X] OBSERVATIONS WHERE CERTAIN CLAIMS WERE FOUND UNSEARCHABLE [1]

This international search report has not been established in respect of certain claims under Article 17(2) (a) for the following reasons:

1. [X] Claim numbers 1-30, because they relate to subject matter [1] not required to be searched by this Authority, namely:

Scientific and mathematical theories

2. [ ] Claim numbers ...., because they relate to parts of the international application that do not comply with the prescribed requirements to such an extent that no meaningful international search can be carried out [1], specifically:

3. [ ] Claim numbers ____________, because they are dependent claims not drafted in accordance with the second and third sentences of PCT Rule 6.4(a).

VI. [ ] OBSERVATIONS WHERE UNITY OF INVENTION IS LACKING [2]

This International Searching Authority found multiple inventions in this international application as follows:

1. [ ] As all required additional search fees were timely paid by the applicant, this international search report covers all searchable claims of the international application.
2. [ ] As only some of the required additional search fees were timely paid by the applicant, this international search report covers only those claims of the international application for which fees were paid, specifically claims:
3. [ ] No required additional search fees were timely paid by the applicant. Consequently, this international search report is restricted to the invention first mentioned in the claims; it is covered by claim numbers:
4. [ ] As all searchable claims could be searched without effort justifying an additional fee, the International Searching Authority did not invite payment of any additional fee.

Remark on Protest

[ ] The additional search fees were accompanied by applicant's protest.

[ ] No protest accompanied the payment of additional search fees.

PCT

WORLD INTELLECTUAL PROPERTY ORGANIZATION
International Bureau

INTERNATIONAL APPLICATION PUBLISHED UNDER THE PATENT COOPERATION TREATY (PCT)

(51) International Patent Classification 6 : H04N 1/41, 7/26

A1

(11) International Publication Number: WO 96/09718

(43) International Publication Date: 28 March 1996 (28.03.96)

(21) International Application Number: PCT/US95/12050

(22) International Filing Date: 21 September 1995 (21.09.95)

(30) Priority Data:
08/310,731 22 September 1994 (22.09.94) US

(71) Applicant: HOUSTON ADVANCED RESEARCH CENTER [US/US]; 4800 Research Forest Drive, The Woodlands, TX 77381 (US).

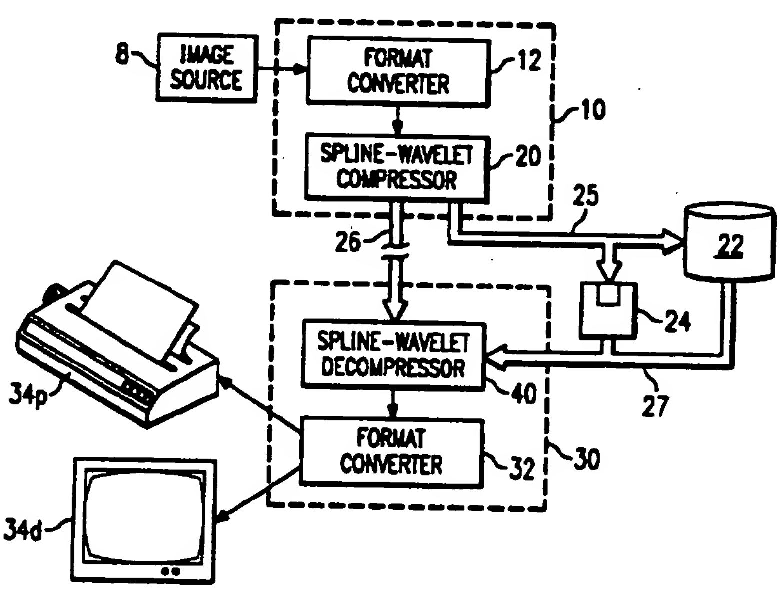
(72) Inventors: CHUI, Charles, K.; 2120 Carter Lake Drive, College Station, TX 77840 (US). YUEN, Pak-Kay; 1528 Hillside Drive, College Station, TX 77840 (US).

(74) Agents: ANDERSON, Rodney, M. et al.; Vinson & Elkins L.L.P., 2500 First City Tower, 1001 Fannin Street, Houston, TX 77002 (US).

(81) Designated States: AM, AT, AU, BB, BG, BR, BY, CA, CH, CN, CZ, DE, DK, EE, ES, FI, GB, GE, HU, IS, JP, KE, KG, KP, KR, KZ, LK, LR, LT, LU, LV, MD, MG, MN, MW, MX, NO, NZ, PL, PT, RO, RU, SD, SE, SG, SI, SK, TJ, TM, TT, UA, UG, UZ, VN, European patent (AT, BE, CH, DE, DK, ES, FR, GB, GR, IE, IT, LU, MC, NL, PT, SE), OAPI patent (BF, BJ, CF, CG, CI, CM, GA, GN, ML, MR, NE, SN, TD, TG), ARIPO patent (KE, MW, SD, SZ, UG).

Published
With international search report.
Before the expiration of the time limit for amending the claims and to be republished in the event of the receipt of amendments.

(54) Title: COMPRESSION AND DECOMPRESSION OF DOCUMENTS AND THE LIKE



(57) Abstract

An apparatus and a corresponding method for performing compression and reconstruction of documents are disclosed. Compression of the document is performed by applying a scaling function and a wavelet function first in one direction (e.g., rows) and then in another direction (e.g., columns) to digital data representing the document. The scaling and wavelet functions correspond to spline and wavelet functions that are compactly supported over a convolution interval, and which can also be implemented by way of integer operations. Byte-packing may also be applied to the image data, without regard to pixel boundaries, allowing not only binary (two-color) documents to be compressed but also enabling the compression of more complex color documents. The spline-wavelet compression and reconstruction may be performed according to alternative methods, including dual-base wavelets, interpolatory wavelets, and wavelet packets.

FOR THE PURPOSES OF INFORMATION ONLY

Codes used to identify States party to the PCT on the front pages of pamphlets publishing international applications under the PCT.

AT	Austria	GB	United Kingdom	MR	Mauritania
AU	Australia	GE	Georgia	MW	Malawi
BB	Barbados	GN	Guinea	NE	Niger
BE	Belgium	GR	Greece	NL	Netherlands
BF	Burkina Faso	HU	Hungary	NO	Norway
BG	Bulgaria	IE	Ireland	NZ	New Zealand
BJ	Benin	IT	Italy	PL	Poland
BR	Brazil	JP	Japan	PT	Portugal
BY	Belarus	KE	Kenya	RO	Romania
CA	Canada	KG	Kyrgystan	RU	Russian Federation
CF	Central African Republic	KP	Democratic People's Republic of Korea	SD	Sudan
CG	Congo	KR	Republic of Korea	SE	Sweden
CH	Switzerland	KZ	Kazakhstan	SI	Slovenia
CI	Côte d'Ivoire	LI	Liechtenstein	SK	Slovakia
CM	Cameroon	LK	Sri Lanka	SN	Senegal
CN	China	LU	Luxembourg	TD	Chad
CS	Czechoslovakia	LV	Latvia	TG	Togo
CZ	Czech Republic	MC	Monaco	TJ	Tajikistan
DE	Germany	MD	Republic of Moldova	TT	Trinidad and Tobago
DK	Denmark	MG	Madagascar	UA	Ukraine
ES	Spain	ML	Mali	US	United States of America
FI	Finland	MN	Mongolia	UZ	Uzbekistan
FR	France			VN	Viet Nam
GA	Gabon				

COMPRESSION AND DECOMPRESSION OF DOCUMENTS AND THE LIKE

TECHNICAL FIELD OF THE INVENTION

This invention is in the field of data compression, and more specifically is in the field of data compression of documents and similar images that may be represented as digital data.

BACKGROUND OF THE INVENTION

The increase in computational speed achieved by modern computer technology has enabled the widespread use of electronic techniques for the communication and storage of documents of many types. For purposes of this application, the term "documents" will refer to two-dimensional representations or renderings of a nature that are conventionally made on a paper medium or on a graphics display by writing, printing, typing, drawing, operation of a computer-aided draw, paint or design program, or other similar conventional techniques, and those renderings that are commonly stored on paper, microfilm, microfiche, or electronically. Examples of such documents include, without limitation thereto, text documents, bank checks and other banking transaction records, vital records, maps, charts, printed works (including combinations of text and graphics), seismic plots, medical records, bank and insurance records, directories, and the like. As is well known to the public, examples of conventional electronic communication and storage of documents includes modern digital

facsimile ("fax") equipment, CD-ROM storage and distribution of encyclopedias and other series of books, electronically or magnetically stored representations of transactions such as checks and other banking transactions and statements, and the like.

Electronic communication and storage is made quite difficult for those documents in which a significant portion of the information contained in many documents is in the form of graphic information which cannot readily be coded into computer-readable form on a character-by-character basis. For example, while the letters in a person's signature may readily be electronically communicated by way of an ASCII coded representation on a character-by-character basis, the retrieval of the ASCII coded representation will not provide any information regarding the appearance of the signature as made by the person. To electronically communicate or store documents with graphic information, the conventional technique is to digitize the document into a bit-map representation, with each memory location in the bit map corresponding to an elemental position of the document, and with the contents of the memory locations corresponding to the color at that location. In the case of a "binary" digital document representations, for example, the digital value "0" may be used to represent the color black and the digital value "1" may be used to represent the color white.

Of course, it is highly desirable that the digitizing of documents be performed with as high a resolution as possible, which requires the use of a significant amount of memory for each document page. For example, current high resolution digitization devotes on the order of greater than one million bits to store a binary bit map representation of a single monochromatic

(black and white) document page divided into 1024-by-1024 picture elements, or "pixels". On a 300 dpi laser printer, this 1k-by-1k picture will be printed on a paper area of approximately three inches by three inches. Therefore, even finer resolution may be required for larger high quality pictures to be printed, and also for certain types of documents, such as fingerprint files utilized by law enforcement agencies.

These high resolution digitally stored representations of documents obviously not only occupy a great deal of computer memory and hard disk storage space, but also require significant time for their transmission over conventional communication lines. As such, conventional data compression techniques are highly useful in the electronic communication and storage of documents. Conventional data compression techniques are generally referred to as of either "lossless" or "lossy", depending upon whether data is discarded in the compression process. For most digitized documents with graphics information, lossy data compression techniques may be used, so long as the receiver is still able to distinguish the graphic image with acceptable clarity.

A survey of conventional lossy data compression techniques may be found at Simon, "How Lossy Data Compression Shrinks Image Files", PC Magazine (July 1993), pp. 371 et seq. A popular one of these conventional lossy data compression techniques is referred to as the JPEG (Joint Photographic Experts Group) method. A description of this technique may be found in Barnsley and Hurd, Fractal Image Compression (AK Peters, Ltd., 1993), pp. 219-228. The JPEG compression method initially divides the image into blocks of pixels, and a Discrete Cosine Transform (DCT) is performed on each pixel block, producing a representation of the block

as coefficients corresponding to frequencies and amplitudes, rather than corresponding directly to color information. These coefficients are then quantized, or rounded off, and a difference algorithm is performed over all quantized blocks in the image, in a selected scan order. This difference algorithm subtracts a DC term corresponding to the mean pixel value of a block, from the DC term of the preceding block. The difference coefficients are then scanned in a different order, such as a zig-zag order, and the non-zero coefficients (i.e., blocks in which a difference from the preceding block occurred) are coded to indicate the number of preceding zero coefficients (i.e., the number of pixel blocks in which no change occurred) and also the value of the non-zero difference. Lossless compression is then often applied to the coded result to further compress the data. Decompression is performed by reversing the compression process, producing the displayable image.

Another conventional method of lossy video image compression, referred to as Recursive Vector Quantization (RVQ), quantizes the pixel blocks directly, without a DCT or other transform, according to a set of selected reference tiles. <u>See</u> Simon, July 1993, <u>op. cit.</u>. The reference tiles are selected according to an iterative technique, based upon the accuracy of the results relative to the original image. As noted in the Simon article, compression according to the RVQ method is computationally intense and complex, but decompression can be done quite rapidly.

Another type of conventional lossy video image compression techniques is referred to as fractal compression. As is well known in the art, a fractal is a mathematical image object that is self-similar, in that the image can be represented in terms of other pieces of

the image. In fractal image compression, the input image is similarly divided into pixel groups, or tiles. Each tile is then approximated by a transformation (contractive, rotational, or both) of one or more other reference regions of the image. The compressed image thus consists of a full representation of the reference region, plus the transformation operators for each of the tiles. Each tile of the image is decompressed by performing a transformation of the reference region using the stored transformation operator for that tile. Detailed descriptions of conventional fractal image compression techniques and systems for performing the same may be found in Barnsley & Hurd, Fractal Image Compression (AK Peters, Ltd., 1993), in U.S. Patent No. 4,941,193, and in U.S. Patent No. 5,065,447.

Wavelet-based techniques are also known to be used for the data compression of digitally stored documents. An summary of this technique, as applied to the compression of digitized fingerprint files used in law enforcement, is described in Bradley and Brislawn, "The Wavelet/Scalar Quantization Compression Standard for Digital Fingerprint Images", Tec. Rep. LA-UR-94-827, Proc. IEEE ISCAS-94 (IEEE, 1994). This approach utilizes the well-known Daubechies wavelet, applied to the document data by way of a table look-up technique. The Daubechies wavelet function used in this case is not bounded, however, and as such boundary effects may be present in the reconstruction; indeed, the approach described in the Bradley and Brislawn paper attempts to reduce boundary effects by reflecting the wavelet filters at the image boundaries. Furthermore, this prior approach is generalized to 8-bit images (i.e., for compression of full grey scale images). Use of this approach for documents, particularly for binary documents, will thus be especially cumbersome considering

the size of the array to be considered in the image compression process, and the eventual compression ratio will be somewhat limited.

By way of further background, the technique of byte packing as a lossless compression technique is well known in the field of computer data storage. According to this well-known technique, digital values that may be expressed by fewer bits are packed with other similar values into a single byte. For example, if eight bit words are used to express a stream of data having only two values, e.g., 255_2 and 0_2, byte packing will allow the data stream to be expressed with a single bit per value. Each byte (eight bits) will thus be able to express the same information as eight bytes of raw data.

By way of further background, our copending application S.N. 08/181,663, filed January 14, 1994, entitled "Method and Apparatus for Video Image Compression and Decompression Using Boundary-Spline-Wavelets", assigned to Houston Advanced Research Center and incorporated herein by this reference, describes a wavelet analysis-based approach to video compression, particularly in the field of motion pictures. This application S.N. 08/181,663 also provides a discussion of the background theory of wavelet analysis, and its application to image compression.

It is an object of the present invention to provide a method and apparatus for compressing digitized representation of documents to a high degree, in such a manner that the resolution and fidelity of the received document upon decompression is maintained.

It is a further object of the present invention to provide such a method and apparatus in which the

decompressed document may be magnified to an arbitrary extent.

It is a further object of the present invention to provide such a method and apparatus in which boundary effects are effectively eliminated in the compressed document, thus improving the quality of the compressed document and also enabling zoom-in and zoom-out techniques to be applied to the reconstructed document.

It is a further object of the present invention to provide such a method and apparatus in which the compression may be performed by integer operations, increasing the speed of the compression and enabling relatively low cost computing equipment (or hardware) to perform the compression quickly.

It is a further object of the present invention to provide such a method and apparatus that is suitable for storage and communication of compressed digitized documents.

It is another object of the present invention to provide such a method and apparatus in which the compression technique is selectable.

Other objects and advantages of the present invention will be apparent to those of ordinary skill in the art having reference to the following specification together with its drawings.